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poświęcone są publikacjom naukowym z zakresu teorii i badań doświadczalnych w dziedzinie mechaniki i termodynamiki przepływów, ze szczególnym uwzględnieniem problematyki maszyn przepływowych

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ZYGMUNT WIERCIŃSKI[1]

The stochastic theory of the natural laminar-turbulent transition in the boundary layer

The aim of the paper is to present in a systematic way the stochastic theory of the natural laminar-turbulent transition in the boundary layer. A new description is put forward of the intermittency factor by means of the three-parameter cumulative function of the Weibull distribution. Transforming this function, it is possible to obtain a linear equation allowing us to determine the point of onset of laminar-turbulent transition, the characteristic length of the transition and the rate of turbulent spot production g_0. Although the intermittency factor calculated theoretically based on discontinuous functions for the rate of spot production (Dirac delta and step function) agrees well with the experimental values, there is a need for search of a continuous left-bounded function corresponding more appropriately to experimental observations.

1. Introduction

Experimental observations show that the process of laminar-turbulent transition is not marked by an abrupt change in the state of the boundary layer. It is a process which takes places in several stages and is preceded by Tollmien-Schlichting waves. It is believed that the flow instabilities develop from the Tollmien-Schlichting wave to three-dimensional amplification of disturbances, followed by the transverse development of peaks and valleys, ejection processes and the formation of turbulent spot (TS). The turbulent spots merge while convecting downstream and make up a fully turbulent flow. Fig. 1 shows schematically subsequent stages in the development from the laminar to turbulent flow in a boundary layer.

The appearance of the Tollmien-Schlichting waves is a consequence of amplification of disturbances in the laminar boundary layer. This process is relatively well described by the Orr-Sommerfeld equation whose solution agrees well with the experimental results. However, the description of the subsequent stages of development of the disturbances is more complicated, as due to increasing nonlinearity the disturbances can not be subject to the rigorous global analysis. Certain methods of fragmentary analysis are possible, however they give an incomplete picture of the process of laminar-turbulent transition. A reasonably sound idea is

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to apply a fenomenological theory. In the present paper this theory will be applied to explain the process of formation of turbulent spots in the boundary layer.

Besides the location of the laminar-turbulent transition, including the beginning and the end of the region, it is the intermittency factor which plays an essential role in the stochastic theory of the laminar-turbulent transition. Given the distribution of the intermittency factor in the region of laminar-turbulent transition, it is possible to evaluate other parameters of the flow in the boundary layer, for example the local drag coefficient C_f. Therefore, the intermittency factor is a convenient parameter for the description of the laminar-turbulent transition in the boundary layer.

The paper is thought to give a systematic presentation of the stochastic theory of the natural laminar-turbulent transition in the boundary layer. The intermittency factor in the transition region will be described by means of the Weibull cumulative distribution function. Based on this distribution the point of onset of laminar-turbulent transition, the characteristic length of the transition as well as the rate of production of turbulent spots per unit area and time will also be determined.

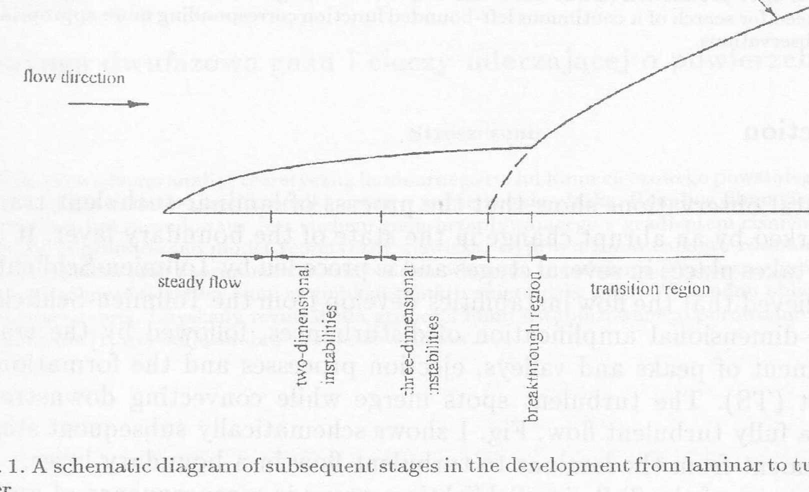

Fig. 1. A schematic diagram of subsequent stages in the development from laminar to turbulent boundary layer.

2. The stochastic theory of natural laminar-turbulent transition in the boundary layer

Turbulent spots in the laminar-turbulent transition were first observed in 1951 by H. W. Emmons from the Harvard University. This phenomenon was observed on a hydrodynamic similarity test rig set up originally for investigating wavy phenomena in supersonic flows. The experiment revealed that the region of laminar-turbulent transition has a structure of turbulent spots. The shape of the turbulent spots is not regular, however bearing much resemblance to an arrow head or an isosceles triangle with a deformed base and rounded vertices whose one vertex overlooks downstream. Emmons also observed that the turbulent spots form in a random way, then convect downstream, grow, come into contact with other turbulent spots and merge. At the end of the region of laminar-turbulent transition, the already merged turbulent spots form a fully turbulent boundary layer. The idealised shape of a turbulent spot is shown in Fig. 2.

Making use of the probability calculus, Emmons set forth a stochastic theory of the laminar-turbulent transition. In order to give a qualitative description of the laminar-turbulent transition, Emmons introduced a source rate density $g(x, y, t)$ which defines the process of spontaneous production of turbulent spots in a unit area and time. At the lowest level of simplification, Emmons did not impose any restrictions on the function $g(x, y, t)$, save for the fact that the function is non-negative. As the convecting turbulent spots grow and increase their cross-section area, let us define a propagation cone with an origin in $P_0(x_0, y_0, t_0)$, see Fig. 3, which is the point of emergence of the turbulent spot at the surface of the body submerged in the flow. Fig. 3, as drawn in the paper of Emmons, presents the propagation cone and its projection on $x-y$ plane.

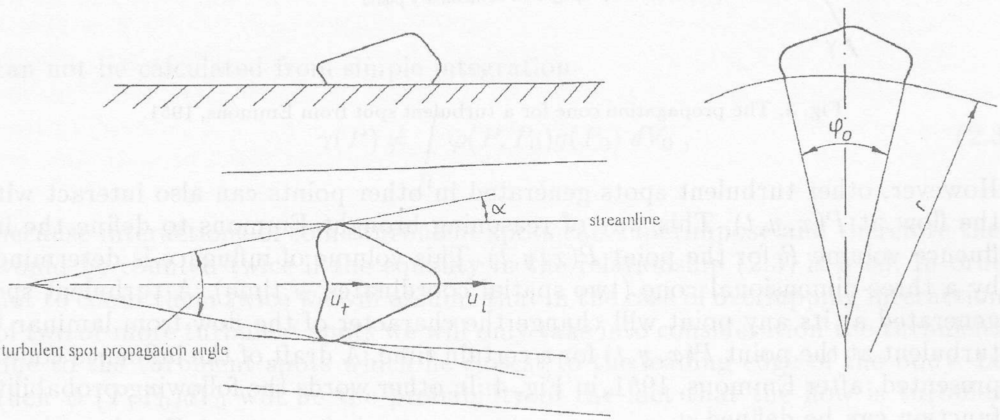

Fig. 2. The idealised shape of a turbulent spot from Emmonds, 1951.

Let us now observe the flow from another point of reference $P(x, y, t)$ located at the body surface. The essence of the observations is measurement of a time interval within which the flow above the point $P(x, y, t)$ is turbulent. This way Emmons introduced, probably for the first time, the notion of an intermittency factor defined as a ratio of time when the flow is turbulent above the point $P(x, y, t)$ to the total time of observations. The turbulent spot generated at the point $P_0(x_0, y_0, t_0)$ will have an impact on the flow at $P(x, y, t)$ if the point $P(x, y, t)$ lies within the propagation cone originating in $P_0(x_0, y_0, t_0)$.

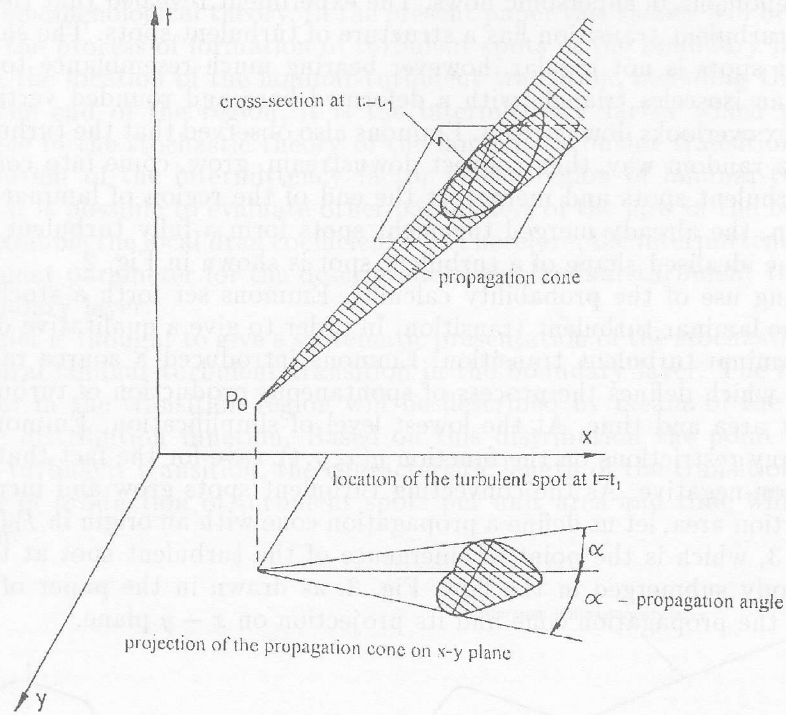

Fig. 3. The propagation cone for a turbulent spot from Emmons, 1951.

However, other turbulent spots generated in other points can also interact with the flow at $P(x, y, t)$. This way of reasoning brought Emmons to define the influence volume R for the point $P(x, y, t)$. This volume of influence is determined by a three-dimensional cone (two spatial coordinates + time). A turbulent spot generated at its any point will change the character of the flow from laminar to turbulent at the point $P(x, y, t)$ for a certain time. A draft of the influence cone is presented, after Emmons, 1951, in Fig. 4. In other words the following probability function can be defined

$$\varphi(P, P_0) = \begin{cases} 1, & P_0 \text{ belongs to the influence volume} \\ 0, & P_0 \text{ does not belong to the influence volume.} \end{cases} \tag{2.1}$$

Given the source rate density $g(x, y, t)$, the number of turbulent spots in an infinitesimal volume can be written as follows

$$g(P_0)dx_0dy_0\,dt\,. \tag{2.2}$$

Each of these turbulent spots will induce turbulent flow at the point P with the probability $\varphi(P, P_0)$. However, the time interval during which the flow is turbulent

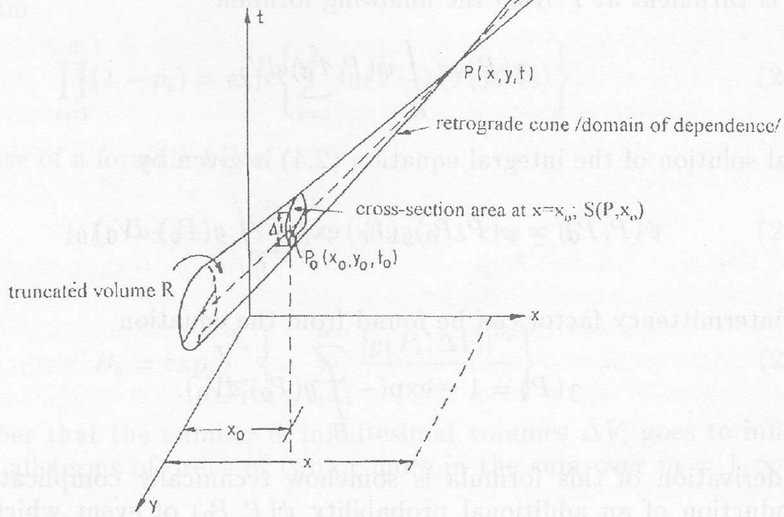

Fig. 4. The influence volume for the point $P(x, y, t)$ from Emmons, 1951.

can not be calculated from simple integration

$$\gamma(P) \neq \int_R \varphi(P, P_0) g(P_0)\, dV_0 \,, \tag{2.3}$$

because interactions of some turbulent spots can superimpose and therefore they would be counted twice if the equality in the relationship (2.3) is used. In order not to count them twice we will assume that in the case of overlapping interactions of two or more turbulent spots we will only take into consideration the turbulence due to the turbulent spots which lie closest to the leading edge of the body. Let then $\psi\,(P, P_0) dV_0$ will be the probability of the fact that the flow is turbulent at the point P due to turbulent spots whose sources lie at points P_0 within the influence cone dV_0, but not at points P_0' for which the condition $x_0' < x_0$ is fulfilled. In other words, the source of a turbulent spot will count as a cause of turbulence at the point P only if the point P_0 is the closest to the leading edge. This probability $\psi(P, P_0)\, dV_0$ can be expressed as below

$$\psi(P, P_0) = \varphi(P, P_0) g(P_0)\, dV_0 \Big[1 - \int_{R'} \psi(P, P_0')\, dV_0'\Big] \,, \tag{2.4}$$

where R' is the part of the influence cone for the point P lying between the leading edge $x = 0$ and the point $x = x_0$. It can be seen that the above equation is an integral equation with respect to $\psi(P, P_0)$.

After calculating the probability $\psi(P, P_0)$, one can find the probability that the flow is turbulent at P from the following formula

$$\gamma(P) = \int\limits_R \psi(P, P_0)\, dV_0. \tag{2.5}$$

A general solution of the integral equation (2.4) is given by

$$\psi(P, P_0) = \varphi(P, P_0) g(P_0) \exp(-\int\limits_{R'} g(P_0)\, dV_0) \tag{2.6}$$

and the intermittency factor can be found from the equation

$$\gamma(P) = 1 - \exp(-\int\limits_R g(P_0)\, dV_0). \tag{2.7}$$

The derivation of this formula is somehow technically complicated due to the introduction of an additional probability $\psi(P, P_0)$ of event which excludes turbulent spots overlapping.

In 1955, J.A. Steketee presented a simpler way of derivation of this formula with the help of the probability theory, assuming n independent events. As an event we will understand the emergence of a turbulent spot in the flow in the boundary layer. In this sense the laminar-turbulent transition will be a stochastic process where the random variables are of binomial distribution with uneven probability of events. Let the probability of an event k is p_k. The probability that the event k will not occur is $1 - p_k$. We will assume that these probabilities are constant but do not have to be equal. The events are not exclusive mutually. In the case of even probability for the emergence of turbulent spots we will have the random variable of simple binomial distribution.

Let us now find the probability $p_{1,n}$ that one out of n events will occur. It is given by the formula

$$p_{1,n} = 1 - \prod_{i=1}^{n} (1 - p_i). \tag{2.8}$$

Assuming that the probability of emergence of a turbulent spot can be decomposed

$$p_i = g(P_i)\Delta V_i, \tag{2.9}$$

what can be achieved by chopping the influence cone into infinitesimal volumes ΔV_i which include points P_i, we can arrive, after Steketee 1955, at the following formula

$$B_n = \prod_{i=1}^{i} (1 - p_i) = \prod_{i=1}^{n} [1 - g(P_i)\Delta V_i]. \tag{2.10}$$

It should be noted that each event (the emergence of a turbulent spot) has been endowed with the infinitesimal volume ΔV_i.

By applying an exponential function, the product in Eq. (2.10) can be substituted with a sum

$$\prod_{i=1}^{n}(1-p_i) = \exp\left\{\sum_{i=1}^{n}\ln(1-g(P_i)\Delta V_i)\right\}. \tag{2.11}$$

Then making use of a formula below

$$\ln(1-x) = -\sum_{1}^{\infty}\frac{x^k}{k}, \quad \text{dla} \quad -1 < x \leq 1 \tag{2.12}$$

we can obtain

$$B_b = \exp\sum_{i=1}^{n}\left\{-\sum_{m=1}^{\infty}\frac{[g(P_i)\Delta V_i]^m}{m}\right\}. \tag{2.13}$$

Assuming further that the number of infinitesimal volumes ΔV_i goes to infinity and neglecting all terms of order of two or more in the sum over $m = 1, \infty$, we can have

$$\sum_{i=1}^{n}\left\{-\sum_{m=1}^{\infty}\frac{[g(P_i)\Delta V_i]^m}{m}\right\} = -\int_V g(P_0)\, dV_0. \tag{2.14}$$

Eventually, if we assign the influence volume V to the point P and substitute the probability of emergence of at least one turbulent spot within the influence cone with the intermittency factor $\gamma(P_0)$, then we obtain

$$\gamma(P) = 1 - \exp\left\{-\int_V g(P_0)\, dV_0\right\}. \tag{2.15}$$

The analysis of this formula leads to the conclusion that in order to evaluate, based on this theory, the intermittency factor, it is necessary to have knowledge of the source rate density $g(x, y, t)$ as well as of the influence cone R and volume dV_0.

It is also worth underlining here that the domain of the function $g(x, y, t)$ does not have to be identical with the geometric dimensions of the propagation cone nor with the influence cone. The beginning of the region of laminar-turbulent transition can locally be defined as the point of emergence of a turbulent spot, that is the beginning of the propagation cone for this turbulent spot. Also the extension of the influence cone R can not range beyond the point of emergence of the first turbulence spot. In other words, the base of the influence cone is located at the point of emergence of the first turbulence spot. Let us denote the point of emergence of the first turbulence spot as x_B. This point can be thought as the beginning of the laminar-turbulent transition. From this point downstream the intermittency factor γ will for the first time exceed zero. The moment when the intermittency factor reaches unity signals the end of the laminar-turbulent transition and indicates the stochastic merging of two neighbouring turbulent spots.

3. Source rate density

According to Emmons, 1951, it is not very likely that a suitable theory to provide the description of the rate of turbulent spot production will be worked out based solely on theoretical investigations, without experimental research. For a mean steady flow in the boundary layer, the rate of spot production, also referred to as the source rate density, denoted as $g(x,y,t)$, can be assumed time-independent, that is

$$g(x,y,t) = g(x,y). \tag{3.1}$$

In his considerations over the laminar-turbulent transition on a flat plate, H.W. Emmons, 1951, assumed that the function g is constant from the leading edge to the trailing edge, however explaining that this function is nonzero downstream of the beginning of the laminar-turbulent transition. This function can be written as follows

$$g(x,y) = \begin{cases} g & \text{for} \quad x \geq x_B, \\ 0 & \text{for} \quad x < x_B. \end{cases} \tag{3.2}$$

Standing on the grounds of experimental investigations of Schubauer & Klebanoff, 1955, which provide evidence that the turbulent spots emerge in a certain narrow breakthrough region at the beginning of the transition region, S. Dhavan & R. Narasimha, 1958, proposed the application of the Dirac delta distribution to approximate the source rate density. The Dirac delta can be written as a limit of the sequence of the Gauss distribution functions

$$g_n(x) = \frac{N}{\sqrt{\pi}} \exp[-N^2(x-x_B)^2], \tag{3.3}$$

where x_B denotes the coordinate x of the beginning of the laminar-turbulent transition and N is a sequence index.

As a supplement to our considerations over the source rate density g, we can propose that this function can increase in a linear manner within an interval between x_B and x_{gE}, where x_{gE} may or may not necessarily be the end of the laminar-turbulent transition. Then, the generation function will have the form

$$g(x) = \left\{ \begin{array}{cl} 0 & \text{for} \quad x < x_B, \\ g_0(1-(x_{gE}-x)/(x_{gE}-x_B)) & \text{for} \quad x_B \leq x \leq x_{gE}, \\ g_0 & \text{for} \quad x > x_{gE}. \end{array} \right\} \tag{3.4}$$

The source rate density g should be different from zero downstream of x_{gE}, otherwise the intermittency factor would also be zero there.

It is also possible to introduce another continuous distributions of the function $g(x)$. These distributions ought to be left-bounded and nonzero values of g should not extend upstream of the beginning of the laminar-turbulent transition x_B. Amongst the possible distributions, one can think of, are the gamma, Maxwell or Rayleigh distributions. These continuous and left-bounded functions represent better the process of formation of turbulent spots and the shape of the function g.

It should be noted here that all the above mentioned distributions assume the two-dimensional model of the boundary layer, that is the two-dimensional model of the laminar-turbulent transition.

4. Influence volume (cone)

The definitions in Chapter 2 of the propagation cone and then the influence volume (cone) R for the point P have a fundamental importance for the elaboration of the stochastic theory of the laminar-turbulent transition.

H.W. Emmons, 1951, made an assumption that each point of the turbulent spot moves along a straight line with a constant velocity. Therefore, the propagation cone is a geometrical cone. Also the influence cone is a cone in the geometrical sense although it does not have to be a cylindrical cone. Under these assumptions and assuming that the generation function is constant one can carry out integration in the intermittency formula to obtain

$$\gamma(V) = 1 - \exp(-gV). \tag{4.1}$$

The volume of the influence cone V can be found by multiplying the base area of the cone by one third of its height

$$V = A_b x/3. \tag{4.2}$$

Moreover, if we assume that the cross-section area of the cone at a unit distance from the vertex of the cone is A_l, then we have

$$A_b = A_l x^2, \tag{4.3}$$

that is subsequent cross-section areas of the influence cone are proportional to the distance between them in second power.

In his calculations Emmons assumed that the unit cross-section of the influence cone is inversely proportional to the velocity U at the top of the boundary layer and proportional to a dimensionless parameter which describes the propagation of the turbulent spots σ

$$A_l = \sigma/U. \tag{4.4}$$

Placing Eqs. (4.3) and (4.4) into (4.2) one can find the expression to evaluate the volume of the influence cone

$$V = \frac{\sigma}{U}\frac{x^3}{3}. \tag{4.5}$$

According to Emmons the parameter depends on the geometry and kinematics of the turbulent spots

$$\sigma = \alpha_p^2 \lambda/\beta, \tag{4.6}$$

where $\alpha_p = \operatorname{tg}\alpha = 0.17$ (the experimental data of Emmons provide the suggestion that $\alpha = 9.6^o$), $\beta = U_{cs}/U$, where U_{cs} denotes the velocity of the turbulent spot centre, whereas U is the velocity outside the boundary layer, (the experimental data of Emmons provide that $\beta = 0.46$), $\lambda = S_s/0.5h^2 = 2$, S_s – cross-section area of the turbulent spot, h – its width.

The dimensionless propagation parameter is also called the Emmons parameter. Based on the above formulas its value can be estimated at $\sigma = 0.1$.

K. K. Chen & N. A. Thyson, 1971, provided a general formula for the cross-section area of the influence cone. They adopted the assumption that if the cross-section area of the turbulent spot has the shape of an isosceles triangle of rounded vertices (arrow head), then also the shape of the cross-section of the influence cone can be assumed similar. They considered the problem in space(two-dimensional)-time domain of s, ϕ, t, where s and $\int r d\phi$ are coordinates at the body surface and t is time. They assumed that the turbulent spot is generated at the point $P_0(s_0, \phi_0, t_0)$ and its shape depends on the way it moves and the velocity of its motion. If the vertex A, or the leading edge of the turbulent spot is convected with the velocity U_l, then the time increme is equal to

$$t - t_0 = \int\limits_{s_0}^{s} \frac{ds}{U_l}. \tag{4.7}$$

The vertices B and C, or the trailing edges not only moved downstream by a distance of

$$t - t_0 = \int\limits_{s_0}^{s} \frac{ds}{U_t}, \tag{4.8}$$

but also moved aside (to increase the width of the turbulent spot), that is

$$\varphi - \varphi_0 = \int\limits_{s_0}^{s} \frac{\operatorname{tg}\alpha ds}{r}. \tag{4.9}$$

The cross-section area of the turbulent spot at the point s_0 is given by the following formula

$$A = r(s_0) \int\limits_{s_0}^{s} \frac{\operatorname{tg}\alpha ds}{r} \int\limits_{s_0}^{s} \left(\frac{1}{u_t} - \frac{1}{u_l}\right) ds. \tag{4.10}$$

The influence cone R can be therefore constructed at any axisymmetrical surface by changing constants and variables in the equations describing the motion of the vertices of the turbulent spot, that is its leading and trailing edges, and reversing the course of time.

5. The intermittency factor

Having determined the source rate density g and then the shape of the influence cone R, we can now return to our considerations concerning the intermittency factor γ. First of all let us recall the formula for the intermittency factor given in the work of Emmons, 1951. Placing Eq. (4.5) into (4.1) we obtain

$$\gamma(x) = 1 - \exp\left(\frac{\sigma g x^3}{3U}\right) \tag{5.1}$$

Let us also recall the assumptions made by H.W. Emmons concerning the source rate density g and the influence cone R. Emmons assumed that the function g is practically nonzero from the leading edge of the plate and remains constant over the full length of the plate. The influence cone has the shape of a geometrical cone of linear generatrix as the experiments provide evidence that the edges of the turbulent spots move along straight lines. The dimensionless propagation parameter, or the Emmons parameter σ depends on the geometry and kinematics of the turbulent spot and is equal to $\sigma \approx 0.1$.

In his theory, besides the two probability functions $\varphi(P, P_0)$ and $\psi(P, P_0)$ formulated earlier in the paper, Emmons put forward expressions to calculate the number N of turbulent spots convecting above the measuring point and also introduced the average time T of passing of the turbulent spot above this measuring point. However, it seems that the notions of N and T have not played any important role for the development of the stochastic theory of the laminar-turbulent transition.

The formula for the intermittency factor can be transformed by introducing the Reynolds number Re_x.

$$\gamma(x) = 1 - \exp\left(\frac{\sigma g v^2}{U^4} Re_x^3\right). \tag{5.2}$$

The expression $\sigma g v^3/U^4$ was named by Emmons as the transition parameter. It is worth mentioning that the formula for the intermittency factor exhibits dependence on the coordinate x or Reynolds number in third power.

A more general analysis concerning the shape of the influence cone presented by Chen and Thyson, 1971, yields the following formula for the intermittency factor at the axisymmetrical surface

$$\gamma(s) = 1 - \exp\left\{-\int_{s_0}^{s} g(s')r(s')\left[\int_{s_0}^{s} \frac{\operatorname{tg}\alpha\, ds}{r}\right]\left[\int_{s_0}^{s}\left(\frac{1}{u_t} - \frac{1}{u_l}\right) ds\right] ds'\right\}. \tag{5.3}$$

The experimental research conducted by Schubauer & Klebanoff, 1955, for the subsonic boundary layer at the flat plate as well as the investigations of Chen

& Thyson, 1971, for the supersonic pipe flow indicate that the angle of propagation of the turbulent spot and velocity ratios U_l/U_e and U_t/U_e remain constant. Therefore, Eq. (5.3) can be transformed into the form

$$\gamma(s) = 1 - \exp\left\{ -\sigma \int_{s_0}^{s} g(s')r(s') \left[\int_{s_0}^{s} \frac{ds}{r} \int_{s_0}^{s} \frac{ds}{U_e} \right] ds' \right\}, \tag{5.4}$$

where the Emmons parameter can be expressed as follows

$$\sigma = U_e \left(\frac{1}{U_t} - \frac{1}{U_l} \right) \operatorname{tg} \alpha. \tag{5.5}$$

Based on the experimental data we can assume that

$$U_t/U_e = 0.5, \quad U_l/U_e = 0.8 \text{ and} \alpha = 10^o \quad \text{or} \quad \operatorname{tg} \alpha = 0.18 \tag{5.6}$$

and eventually obtain the value of the dimensionless propagation parameter σ

$$\sigma \cong 0.135$$

As seen from the comparison this value is close to that given originally by Emmons.

In the case of a flat plate the formula (5.3) can be simplified into

$$\gamma(s) = 1 - \exp\left[-\frac{\sigma}{U_e} \int_{s_0}^{s} g(s')s'^2 ds' \right]. \tag{5.7}$$

The experiments of Schubauer & Klebanoff, 1955, provide evidence that the turbulent spots appear in the breakthrough region at the beginning of the laminar-turbulent transition. The breakthrough region has a limited extension. According to S. Dhavan & R. Narasimha, 1957, it is well-founded to approximate the source rate density with the Dirac delta distribution

$$g(x) = g_0 \delta(x - x_B), \tag{5.8}$$

where the index B denotes as earlier the beginning of the laminar-turbulent transition and g_0 is the concentration of the turbulent spots in a unit area (length) and time.

Inserting Eq. (5.8) to (5.4) we can easily obtain the formula for the intermittency factor as below

$$\gamma(x) = 1 - \exp\left[-\frac{g_0 \sigma}{U_e} (x - x_B)^2 \right]. \tag{5.9}$$

It is seen that the change in the source rate density from a uniform distribution to the Dirac delta brings about the change in power at the coordinate x from three to two. The dependence on the Reynolds number will change likewise and the following dimensionless transition parameter will be obtained

$$g_0\sigma\frac{(x-x_B)^2}{U_e}=\hat{n}\sigma(Re_x-Re_B)^2, \tag{5.10}$$

where $\hat{n}=g_0v^3/U_e^3$ is the dimensionless turbulent spot production parameter, Mayle, 1991.

Assuming that the function g is the Dirac delta distribution (Eq. (3.3)), Chen & Thyson, 1971, gave formulas for the intermittency factor for the boundary layer in the flow past a cone

$$\gamma(s)=1-\exp\left(n\sigma s_B\frac{s}{U_e}\frac{s_B}{}\ln\frac{s}{s_B}\right) \tag{5.11}$$

and past a hemisphere of radius R

$$\gamma(s)=1-\exp\left(-\frac{n\sigma}{R}\sin\frac{s_B}{R}\ln\frac{\operatorname{tg}\frac{s}{2R}}{\operatorname{tg}\frac{s_B}{R}}\ln\frac{s}{s_B}\frac{1}{\frac{dU}{ds}}\right), \tag{5.12}$$

where dU/ds should be calculated at the stagnation point at the front of the hemisphere.

For a flat plate, assuming a linear distribution of the source rate density between the beginning x_B and the end x_E of the laminar-turbulent transition as in Eq. (3.4), putting there $x_{gE}=x_E$, we will obtain the following formula for the intermittency factor

$$\gamma(x)=1-\exp\left\{-\frac{g_0}{x_E-x_B}\left[\frac{x^4-x_B^4}{4}-x_B\frac{x^3-x_B^3}{3}\right]\right\}. \tag{5.13}$$

It is clear that for the linear distribution from x_B to x_E of the source rate density the exponent in the formula for the intermittency factor exhibits the spatial coordinate x in fourth power.

For a uniform distribution of the function g from the beginning x_B to the end x_E of the laminar-turbulent transition the intermittency factor is equal to

$$\gamma(x)=1-\exp\left[-\frac{g_0\sigma}{3U_e}(x-x_B)^3\right]. \tag{5.14}$$

Putting $x_B=0$, we can obtain Eq. (5.1) derived originally by Emmons.

Below are presented three other expressions for the intermittency factor which can be found in the literature. The first expression will be cited after Klebanoff & Schubauer, 1955, who put forward a universal distribution of the intermittency factor in the form of the Gauss cumulative distribution function

$$\gamma(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-(x-\eta)^2/2\sigma^2}\,dx, \tag{5.15}$$

where
$$\eta = \frac{x - \overline{x}}{\sigma} \tag{5.16}$$

and x denotes the distance from the leading edge, $\overline{x}$ is the median of the distribution. In the experimental investigations of Schubauer & Klebanoff, 1955, values of the parameter σ range from 0.1 to 0.24.

The Gauss cumulative distribution function can be approximated with the help of the following formula (Fraser et al., 1988)

$$\gamma = \frac{1}{2}\left\{1 + \frac{\eta}{|\eta|}\left(0.8273|\eta| - 0.094|\eta|^2 - 0.073|\eta|^3 + 0.0165|\eta|^4\right)\right\}. \tag{5.17}$$

Another proposal of a universal distribution of the intermittency factor γ was presented in the paper of Dhavan & Narasimha, 1958,

$$\gamma(\xi) = 1 - e^{-0.412\xi^2}, \tag{5.19}$$

where
$$\xi = \frac{x - x_E}{\lambda} \quad \text{i} \quad \lambda = x|_{\gamma=0.75} - x|_{\gamma=0.25}. \tag{5.19}$$

The distance λ is used here to normalize the intermittency factor λ and was chosen arbitrarily as a quantile difference in the form of the double quantile deviation, being a measure of diffusion of the intermittency distribution, see Fisz 1958.

A slightly different universal distribution of the intermittency factor γ in the boundary layer can be found in the work of Abu-Ghannam & Shaw, 1980

$$\gamma(\zeta) = 1 - e^{-5\zeta^3}, \tag{5.20}$$

where
$$\zeta = \frac{x - x_B}{x_E - x_B}, \tag{5.21}$$

and x_B, x_E are the beginning and the end, respectively, of the laminar-turbulent transition measured from the leading edge of the plate.

These three formulas presented above, Eqs.(5.17), (5.18) and (5.20), were compared with the experimental investigations in the work of C. Fraser et al., 1988. Although each formula contains a different distribution of the source rate density (normal distribution, Dirac delta and step function), the results agree reasonably well with the experiment in all three cases.

The comparison made by Wierciński, 1995, shows that Eqs. (5.18) and (5.20) give satisfactory agreement with the results of measurements of the intermittency factor at a distance of $y/\delta = 0.1$ from the body. A number of different techniques concerning the measurement of the intermittency factor can be found in the literature, starting from a simple technique which draws on measuring the time of occurrence of high-frequency fluctuations of the velocity, ending up with more sophisticated methods of analysis of fluctuation histograms. It should also be realised that the intermittency factor depends as well on the cross-flow coordinate in the boundary layer, see for example Wierciński, 1995. Therefore, it should not be overly surprising that it is possible to obtain equally satisfactory coincidence between the experiments and theory for different theoretical approaches, based on more or less justified theoretical foundations.

6. The intermittency factor as the cumulative function of a compound generalised Weibull distribution

It has been shown in Chapter 2, after Emmons, 1951, and Steketee, 1955, that the probability of occurrence of a turbulent spot at any point of the region of laminar-turbulent transition can be described by means of a generalized Bernoulli scheme (generalized binomial distribution). As the source rate density is of vital importance for our discussion of probability of occurrence of the turbulent flow at our reference point we have also presented in Chapter 3 some forms of distribution of this function known in the literature, from the step function to the Dirac delta distribution.

The analysis carried out so far leads to the conclusion that the intermittency factor can be described with the help of the cumulative function of a compound generalized Weibull distribution. The description of the Weibull distribution and its application in different engineering fields can be found in Lipson & Sheth, 1973, or Wadsworth, 1990. The probability density and cumulative function of the Weibull distribution are given by the following formula

$$f(x) = \alpha\beta(x - x_0)^{\alpha-1}\exp(-\beta(x - x_0)^{\alpha}), \tag{6.1}$$

$$F(x) = 1 - \exp(-\beta(x - x_0)^{\alpha}). \tag{6.2}$$

The Weibull distribution is a three-parameter probability distribution with the parameters α, β, and x_0. For $\alpha = 1$ it is an exponential distribution, for $\alpha = 2$ it becomes the Rayleigh distribution.

The probability distribution is referred to as compound if its one parameter is a random variable of a certain probability distribution. In our case it is the source rate density g.

The probability distribution is called generalized if its form resembles and is more complicated than the usual distribution it is referred to. We will supply the adjective 'generalized' to the Weibull distribution because its parameters depend on a certain operator, for example integral, of the independent variable of the distribution

$$F(x) = 1 - \exp\left(-\int\limits_{x_0}^{x} h(x')dx'\right), \tag{6.3}$$

$$f(x) = \frac{dF}{dx} = \left(\int\limits_{x_0}^{x} h(x')dx'\right)' \exp\left(-\int\limits_{x_0}^{x} h(x')dx'\right). \tag{6.4}$$

The comparison of Eqs. (2.7) and (2.6) with Eqs. (6.3) and (6.4) clearly shows that the intermittency factor can be described by means of a compound generalized Weibull distribution.

Analysing Eq. (5.1), it can be seen that this is the cumulative function of the Weibull probability distribution with the parameters $\alpha = 3, \beta = g/3U$ and $x_0 = 0$, whereas in Eq. (5.9) we can find a particular form of the Weibull distribution with the parameters $\alpha = 2, \beta = g_0\sigma/U_e$ and $x_0 = x_B$ – a particular form, because for $\alpha = 2$ the Weibull probability distribution becomes the Rayleigh distribution.

A three-parameter cumulative function of the Weibull distribution as the intermittency factor given below in a form which is different from that in Eq. (6.3) can be used to evaluate the parameters of the distribution α, Θ, and x_E

$$\gamma(x) = 1 - \exp\left[-\left(\frac{x - x_B}{\Theta - x_E}\right)^{\alpha}\right], \tag{6.5}$$

where the coefficient β from Eqs. (6.1) and (6.2) is given by

$$\beta = \frac{1}{(\Theta - x_E)^{\alpha}}. \tag{6.6}$$

The coefficient α is called the slope of the Weibull distribution. It follows from the analysis carried out in the previous chapters that for the distributions defined with Eqs. (5.18) and (5.20) this coefficient is $\alpha = 2$ or 3 respectively, or can even be equal to 4 as in Eq. (5.13).

The parameter x_E, called also the location parameter, is the expected minimum of the distribution, in our case it is off course the beginning of the laminar-turbulent transition.

The parameter Θ known as the characteristic value or the scale parameter, see Lipson & Sheth, 1973, can be easily found from the fact that for $x = \Theta$ we have $\gamma(x) = 1 - 1/e = 0.632$.

By transforming and taking logarithm twice of Eq. (6.5)

$$\ln\ln\frac{1}{1-\gamma(x)} = \alpha\ln(x - x_E) - \alpha\ln(\Theta - x_E), \tag{6.7}$$

we can obtain the following linear relationship for evaluating the parameters x_E, α and β, Lipson & Sheth, 1973

$$Y = \alpha X + C, \tag{6.8}$$

where

$$\begin{aligned} Y &= \ln\ln\tfrac{1}{1-\gamma(x)}, \\ X &= \ln(x - x_B), \\ C &= \alpha\ln(\Theta - x_B). \end{aligned} \tag{6.9}$$

This method was validated for the experimental data presented in the paper of Wierciński, 1995, for the case of a flat plate inclined at -2.5^o attack angle to the free stream of mean velocity $U_0 = 15\text{m/s}$. As a result, for the experimental data concerning the intermittency factor $\gamma(x)$, the regression line given by Eq. (6.8) was obtained with the correlation coefficient of $\rho = 0.99$. Then, it was found that the slope of the Weibull distribution is $\alpha = 3.5$, the beginning of the laminar-turbulent transition, $x_0 = 0.226$ m and the scale parameter $\Theta = 0.395$ m. The characteristic length of the laminar-turbulent transition, that is the distance from the beginning of the laminar-turbulent transition to the point where the intermittency factor is $\gamma(x) = 0.632$ is equal to $\Theta - x_0 = 0.169\text{m}$. The coefficient β in Eqs. (6.1) and (6.2) important for evaluating g_0 is $\beta = 1/(\Theta - x_E)^\alpha = 504$. It is worth underlining here that the best agreement with the experiment was obtained for the coefficient $\alpha = 3.5$, that is not for $\alpha = 3$ as in the Emmons equation, nor for $\alpha = 2$ as in the equation of Dhavan & Narasimha. A graph of the function $\ln\ln(1/(1-\gamma(x))$ for four different values of $x_0 = 0.05, 0.135, 0.22$ and 0.305 is presented in Fig. 5. It is apparent from the picture that the best linearity of the relationship (6.8) was reached for $x_0 = 0.22$. In further calculations an even better correlation coefficient was obtained for $x_0 = 0.226\text{m}$.

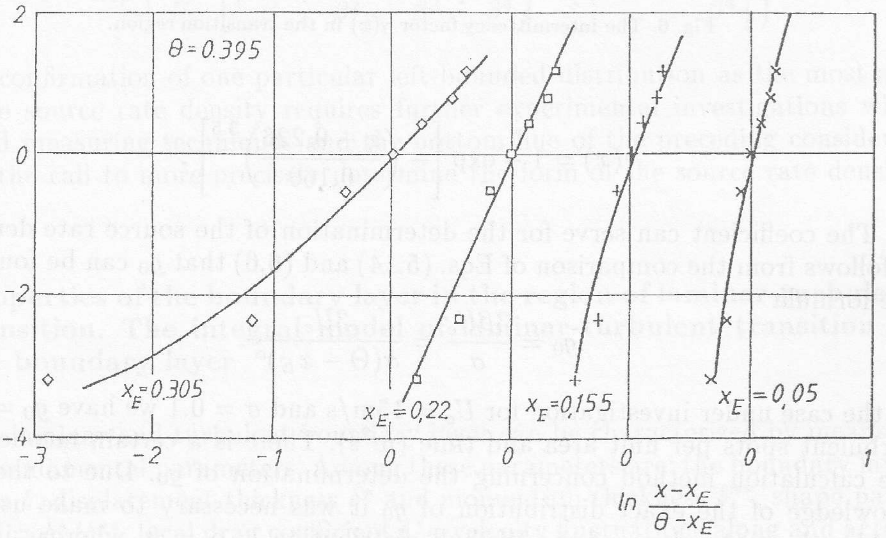

Fig. 5. Searching for the beginning of the laminar-turbulent transition from the Weibull distribution with the help of the linear correlation method.

The results of these calculations are also presented in Fig. 6 where the experimental distribution of the intermittency factor from the paper of Wierciński, 1995, is compared with the theoretical curve plotted based on the formula below

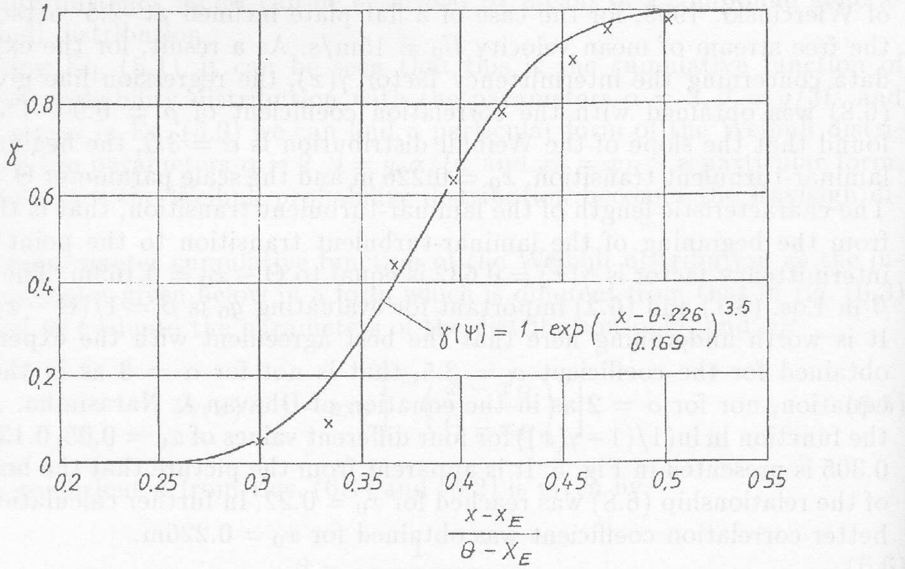

Fig. 6. The intermittency factor $\gamma(x)$ in the transition region.

$$\gamma(x) = 1 - \exp\left[-\left(\frac{x-0.226}{0.169}\right)^{3.5}\right]. \tag{6.10}$$

The coefficient can serve for the determination of the source rate density g_0. It follows from the comparison of Eqs. (5.14) and (6.6) that g_0 can be found from the formula

$$g_0 = \frac{3\beta U_e}{\sigma} = \frac{3U_e}{\sigma(\Theta - x_E)^{\alpha}}. \tag{6.11}$$

In the case under investigation for $U_e = 15\text{m/s}$ and $\sigma = 0.1$ we have $g_0 = 226800$ turbulent spots per unit area and time (m^2s). There is a certain incoherence in the calculation method concerning the determination of g_0. Due to the lack of knowledge of the exact distribution of g_0 it was necessary to make use of Eq. (5.14) where the slope of the Weibull distribution is $\alpha = 3$, whereas it follows from the calculations that $\alpha = 3.5$.

It seems possible as well to apply the distribution proposed by S. Dhavan & R. Narasimha and determine g_0 from Eq. (5.9). Due to the fact that the source rate density is expressed as the Dirac delta, the value of g_0 will be three times less than that given in Eq. (6.11).

The consequence of choice of the Weibull cumulative distribution function to represent the intermittency factor $\gamma(x)$ is the presence of the length parameter $\Theta - x_E$ which determines the distance for which $\gamma(x) = 0.632$. This parameter appears in the cumulative distribution function without referring to the median or quantile difference. Moreover, there should be a simple relationship between this parameter and g_0 similar to that of Eq. (6.10).

As shown in the preceding section, the choice of the Dirac delta or step function as a source rate density $g(x)$ leads to simple formulas for the intermittency factor which generally agree reasonably well with the experimental data. However, attempts to more precisely determine the slope of the Weibull distribution show that it does not have to be an integer – $\alpha = 2$ for the Dirac delta or $\alpha = 3$ for the step function. Anyway the Dirac delta and step function are only first approximations of the source rate density and the experimental observations provide evidence that the function $g(x)$ is neither a constant nor point-type distribution. Assuming the simplest left-bounded distribution - the gamma distribution as the source rate density

$$g(x) = g_0(x - x_B)^{\delta-1} \exp[-\beta(x - x_B)], \tag{6.12}$$

then selecting arbitrarily the value of the parameter $\delta = 2$ and placing the above into Eq. (5.9) we can get the following formula for the distribution of the intermittency factor $\gamma(x)$

$$\gamma(x) = 1 - \exp\left\{\frac{g_0\sigma}{U_e}\left[\left(\frac{x^3}{\beta} + \frac{3x^2}{\beta^2} + \frac{6x}{\beta^3} + \frac{6}{\beta^4}\right)\exp(-\beta x) - \frac{6}{\beta^4}\right]\right\}. \tag{6.13}$$

The confirmation of one particular left-bounded distribution as the most appropriate source rate density requires further experimental investigations with advanced measuring techniques and the bottom line of the preceding considerations is the call to more precisely determine the form of the source rate density $g(x)$.

7. Properties of the boundary layer in the region of laminar-turbulent transition. The integral model of laminar-turbulent transition in the boundary layer

Both laminar and turbulent boundary layer can be characterized by means of several fundamental parameters. Among these parameters are: the boundary layer thickness δ, displacement thickness δ^* and momentum thickness δ^{**}, shape parameter $H = \delta^*/\delta^{**}$, local drag coefficient C_f, velocity fluctuations along and across the boundary layer. Assuming in the integral model of the laminar-turbulent transition that the boundary layer in the region of laminar-turbulent transition is a stochastic mixture of laminar and turbulent boundary layers with a certain value of the intermittency factor as a time ratio of turbulent state to the total observation time, the local mean velocity in the boundary layer can be expressed by the formula

$$\overline{u} = (1-\gamma)\overline{u}_l - \gamma\overline{u}_t \,. \tag{7.1}$$

In 1951, H. W. Emmons gave a relationship to describe the local drag coefficient C_f as a weighted sum of the drag coefficient for the laminar boundary layer C_{fl} with the weight coefficient $(1-\gamma)$ and the drag coefficient for the turbulent boundary layer C_{ft} with the weight coefficient γ

$$C_f = (1-\gamma)C_{fl} - \gamma C_{ft}. \tag{7.2}$$

Combining the formula for the displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{\overline{u}}{\overline{u}_\infty}\right) dy \tag{7.3}$$

into Eq. (7.1) gives

$$\delta^* = (1-\gamma)\delta_l^* - \gamma\delta_t^*. \tag{7.4}$$

The same procedure with respect to the momentum thickness

$$\delta^* = \int_0^\infty \frac{\overline{u}}{\overline{u}_e}\left(1 - \frac{\overline{u}}{\overline{u}_\infty}\right) dy \tag{7.5}$$

yields the following formula, Dhavan & Narasimha, 1958,

$$\delta^{**} = (1-\gamma)[(1-\gamma)\delta_l^{**} - \gamma\delta_t^*] + \gamma[\gamma\delta_t^{**} - (1-\gamma)\delta_t^*] + 2\gamma(1-\gamma)F\,, \tag{7.6}$$

where

$$F = \int_0^\infty \left(1 - \left(\frac{\overline{u}}{\overline{u}_\infty}\right)_l \left(\frac{\overline{u}}{\overline{u}_\infty}\right)_t\right) dy. \tag{7.7}$$

A slightly different formula for the momentum thickness can be found in the work of Gostelov et al., 1994

$$\delta^{**} = \gamma(1-\gamma)\int_0^\infty [u_l(1-u_t) + u_t(1-u_l)] + (1-\gamma)^2\delta_l^{**} + \gamma^2\delta_t^{**}. \tag{7.8}$$

Making use of the above presented relationships for δ^* and δ^{**}, it is easy to determine the shape parameter H for the boundary layer in the region of laminar-turbulent transition (Dhavan, Narasimha, 1958)

$$H = \frac{(1-\gamma)H_l + \gamma H_t \frac{\delta_t^{**}}{\delta_l^{**}}}{(1-\gamma^2)\left[1 - \frac{\gamma}{1-\gamma}H_l\right] + \gamma^2\left[1 - \frac{1-\gamma}{\gamma}H_l\right]\frac{\delta_t^{**}}{\delta_l^{**}} + 2\gamma(1-\gamma)\frac{F}{\delta_l^{**}}}. \tag{7.9}$$

The work of Gostelov et al., 1994, provides also an approximate formula for the shape parameter H in the region of laminar-turbulent transition

$$H = H_t + (1-\gamma)(H_l - H_t). \tag{7.10}$$

The level of velocity fluctuations u', v' and w' in the laminar boundary layer is negligibly small compared to that of the turbulent boundary layer. Therefore, unlike for the mean velocity or shape parameter, it is unpurposeful to attempt to establish the relationship of this type in the transition region for velocity fluctuations.

8. Conclusions

The paper gives a systematic review of the stochastic theory of the natural laminar-turbulent transition in the boundary layer. The choice of the three-parameter cumulative function of the Weibull distribution to describe the intermittency factor enables precise and relatively straightforward determination, from the experimental data, of the coordinate x_0 which defines the beginning of the laminar-turbulent transition in the boundary layer, characteristic length $\Theta - x_0$ of the laminar-turbulent transition, value of the source rate density g_0. Making use of this method the above listed parameters were calculated for the set of experimental data acquired by Wierciński, 1995. It was found that the slope of the Weibull distribution is $\alpha = 3.5$, that is more than for the distribution of the intermittency factor were the source rate density is approximated with the help of the step function – $\alpha = 3$ (Emmons 1951) or Dirac delta – $\alpha = 2$ (Dhavan & Narasimha, 1958). The precise determination of a left-bounded continuous distribution of the source rate density is still a matter for further experimental and theoretical investigations.

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Teoria stochastycznego naturalnego przejścia laminarno-turbulentnego w warstwie przyściennej

Streszczenie

Przedmiotem pracy jest usystematyzowanie teorii stochastycznego naturalnego przejścia laminarno-turbulentnego w warstwie przyściennej. Przedstawiono nową propozycję opisu współczynnika intermitencji przy pomocy dystrybuanty trójparametrowego rozkładu prawdopodobieństwa Weibulla. Przekształcając dystrybuantę rozkładu Weibulla można otrzymać liniowy związek pozwalający na podstawie danych eksperymentalnych na dokładne wyznaczenie początku przejścia laminarno-turbulentnego, a ponadto na określenie długości charakterystycznej przejścia i na określenie wartości g_0 funkcji generacji plamek turbulencji. Nieciągłe rozkłady funkcji generacji plamek turbulencji (delta Diraca i funkcja skoku jednostkowego) pomimo zadawalającej zgodności utworzonych na ich bazie formuł na współczynnik intermitencji z wartościami eksperymentalnymi powinny zostać zastąpione przez rozkłady ciągłe i lewostronnie ograniczone lepiej odpowiadające obserwacjom eksperymentalnym.