P O L S K A   A K A D E M I A   N A U K

INSTYTUT MASZYN PRZEPŁYWOWYCH

# PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH

## TRANSACTIONS OF THE INSTITUTE OF FLUID-FLOW MACHINERY

93

WARSZAWA – POZNAŃ 1992

W Y D A W N I C T W O   N A U K O W E   P W N

PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH

poświęcone są publikacjom naukowym z zakresu teorii i badań doświadczalnych w dziedzinie mechaniki i termodynamiki przepływów, ze szczególnym uwzględnieniem problematyki maszyn przepływowych

*

THE TRANSACTIONS OF THE INSTITUTE OF FLUID-FLOW MACHINERY

exist for the publication of theoretical and experimental investigations of all aspects of the mechanics and thermodynamics of fluid-flow with special reference to fluid-flow machinery



**REDAKCJA – EDITORIAL OFFICE**

Instytut Maszyn Przepływowych PAN
ul. Gen. Józefa Fiszera 14, 80-952 Gdańsk, skr. pocztowa 621, tel. 41-12-71



Printed in Poland

ISBN 83-01-10515-1
ISSN 0079-3205

WYDAWNICTWO NAUKOWE PWN – ODDZIAŁ W POZNANIU

| | |
|---|---|
| Nakład 300+80 egz. | Oddano do składania 14 I 1991 r. |
| Ark. wyd. 17,5. Ark. druk. 15,625 | Podpisano do druku 6 V 1992 r. |
| Pap. offset. kl. III, 70 g 70×100 cm. | Druk ukończono w lipcu 1992 r. |
| Nr zam. 158/187 | |

DRUKARNIA UNIWERSYTETU IM. A. MICKIEWICZA W POZNANIU

PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH

1991 Zeszyt 93

ZYGMUNT WIERCIŃSKI

Gdańsk

# Comparison of the Method of Belotserkovski and the Method of Jacobi Quadratures Applied to the Discrete Vortex Method in Nonlinear Problems of Unsteady Fluid Mechanics

The paper presents a comparison of accuracy of Belotserkovski and Jacobi quadratures method of numerical solution of Cauchy type singular integral equation of the first kind which is the base for a solution of nonlinear problems of unsteady flow about thin profiles by means of discrete vortex approach. The comparison of accuracy of the methods was made for the case of steady flow about a flat plate and the flow about a plate with stationary vortex in the wake. The accuracy of the solution of singular integral equation of Cauchy type with the aid of Gauss- or Radau-Jacobi quadratures is much better than in case of the Belotserkovski's method.

## Notation

$A_i$ – weight coefficients of quadrature,
$C_L$ – lift coefficient,
$C_M$ – moment coefficient,
$F$ – constant,
$g$ – searched function in a quadrature,
$h$ – segment of profile,
$i, j, k$ – integers,
$(l_1, l_2)$ – coordinates of leading and trailing edges of a profile,
$L_P$ – open arc of surface vorticity on a profile,
$L^W$ – open arc of free surface vorticity,
$n$ – normal to profile,
$N$ – number of abscissae and control points, degree of Jacobi polynomial,
$p$ – pressure,
$P_N^{(\alpha,\beta)}(x)$ – Jacobi polynomial of the first kind of the $N$ degree,
$r$ – radius,
$U$ – velocity of fluid,
$v$ – normal component of velocity on a profile surface,
$w(x)$ – weight function,
$w$ – liquid particles motion velocity,
$w_0$ – relative velocity of liquid particles motion,
$z_P=(x, y)$ – point on the profile,
$z_W=(\zeta, \eta)$ – point on free vortex surface,
$x_i$ – abscissa on a flat plate,
$\alpha, \beta$ – parameters of Jacobi polynomials,
$\gamma_P$ – intensity of surface vorticity on a profile,
$\gamma_W$ – intensity of free surface vorticity,
$\Gamma$ – velocity circulation around a profile,
$\Gamma'$ – intensity of a vortex in the wake of a profile,
$\delta$ – angle,
$\Delta$ – difference, relative error,
$\nabla$ – gradient,
$\varphi$ – velocity potential,
$\zeta_1$ – distance of the first discrete vortex at a free vortex surface,
$\mu_i$ – coefficient,
$\nu_k$ – coefficient,

$\nu$ – index of a singular integral equation,
$\Pi_N^{(\alpha,\beta)}(x)$ – Jacobi polynomial of the second kind of the $N$ degree,
$\xi_k$ – control point on a profile.

Suffixes

$-, +$ – refer to upper and lower surface of a profile,
$i$ – refers to $i$-th point,
$k$ – refers to $k$-th point,
$a$ – refers to a known analytic solution.

## 1. Introduction

Methods of surface vorticity on open arcs are often used to solve nonlinear problems of unsteady fluid mechanics. Numerical version of the surface vorticity method is often called the method of discrete vortex. The work [1] is wholly dedicated to solution of the nonlinear problem of unsteady flow about thin profiles by inviscid fluid with the aid of the surface vorticity method. In [2] the surface vorticity method was applied to the description of flow about flat vibrating plate in inviscid fluid. Cauchy type singular integral equation of the first kind (CSIE) is a basic equation for solution of nonlinear problems of unsteady aerodynamics of thin profiles. The methods of solution of CSIE can be divided into indirect and direct ones. The indirect methods consist in transformation of CSIE by regularization process into Fredholm's integral equation or in the application of various approximations of the integrand. Direct methods consist in application of suitable quadratures and this way changing the integral equation directly into a set of linear equations. Accuracy of the direct methods strongly depends on the distribution of the vortex and control points on the open arc.

The attempts to find more precise numerical method of solution of CSIE which can be applied to solution of nonlinear problems of unsteady flow about thin profiles should be considered desirable as the methods used recently, that is the method of uniform distribution of vortices and control points on arc (Belotserkovski's method) and its small modification (Gorelov's method*), is rather inaccurate just in the vicinity of the leading edge and the trailing edge. For a stationary flow about a flat plate the error of determination of the value of vorticity reaches – 11.3% at the leading edge and 2.3% at the trailing edge for the amount of vortices displacement points equal to $N=20$ [9]. According to Gorelov and Kulyaev [10] the accuracy of Belotserkovski's method for $N\to\infty$ equals 11.2% for the first point at the leading edge and 32.9% for the last point at the trailing edge. In this paper the accuracy of Belotserkovski's method [8] and the method of Gorelov [10] for the flow about flat plate with vortex behind the trailing edge is examined. The Jacobi quadratures method is different from the Belotserkovski's method and the Gorelov's method mainly because of the nonuniform distribution of vortices displacemet points and control points along the segment $\langle -1, 1\rangle$.

* The modification consists in displacing the distribution of points applied in the method of Belotserkovski, especially in the vicinity of the leading and trailing edge of a profile.

In the method proposed by Belotserkovski and Nisht [1] the uniform distribution of vortices and control points on arc are used. It is well known that the accuracy of calculation in this method is not the best one [3, 4, 5]. To improve the accuracy of calculations in the vorticity surface method on open arcs the Jacobi quadratures method of CSIE solution is proposed [5]. In this method the distribution of vortices and control points are nonuniform, and are found as roots of appropriate Jacobi polynomials. The Jacobi quadratures were first used by N. I. Ioakimidis and P. S. Theocaris [7] to solve problems of cracking in solid mechanics. The accuracy of the Gauss- and Radau-Jacobi quadratures to solution of a CSIE is by several orders better than the accuracy of the method given in [1, 2]. The accuracy comparison was made in [5] for the stationary flow about flat plats with vortex in its wake, which is a pattern flow of discrete vortex methods for unsteady flows past a thin profile. Wishing to improve the accuracy of their method Belotserkovski and Lifanov have proposed in their new book [8] a nonuniform distribution of vortex and control points by means of Chebyshev quadratures.

## 2. The Nonlinear Problem of Unsteady Flow About a Thin Profile by an Inviscid and Incompressible Fluid

It is well known that the flow of perfect fluid past a thin slightly curved profile can be described by two dimensional equations. Let's introduce the set of coordinates in which fluid far upstream and far downstream of a profile is in motion at a velocity $U$, see Fig. 1. For a slightly curved profile we can assume, without loss of generality, that a profile can be substituted with its chord $x \in \langle l_1, l_2 \rangle$. Most often $l_2=1$, and $l_1=-1$ or $l_1=0$ [8, 10]. What refers to a profile we shall assume that it is in rest or in harmonic motion according to a predetermined rule around a central

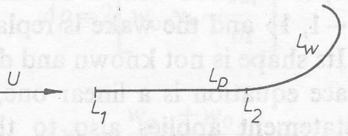


Fig. 1. Thin profile $L_P$ and free surface vorticity $L_W$ in its wake

position (for example, a profile can perform flexural or torsional vibrations). In case of a vibrating profile we assume that the motion started at time $t=0$. We assume then that the motion of fluid about the profile and its wake is potential. In this case, there is a velocity potential and it fulfils the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=0. \tag{1}$$

The boundary conditions for Eq. 1 assume

1) equality of normal components of profile velocity and the velocity of fluid on the surface of a profile $L$, $n$ – normal to the profile

$$(\nabla \varphi - U)\cdot n = 0 \quad \text{for} \quad z_P=(x, y) \in L_P \tag{2}$$

2) vanishing of the velocity disturbances for large distances $r$ from the profile

$$\nabla\varphi\to 0 \quad \text{for} \quad r\to\infty, \tag{3}$$

3) fulfilment of the Kutta-Joukovski condition referring to flow at the edge which leads to the condition of zero pressure difference between two sides of the profile at the trailing point (sign + and − correspond respectively to the upper and bottom surface of the profile),

$$\Delta p = p_+ - p_- \to 0 \quad \text{for} \quad z_P=(x, y)\to l_2, \tag{4}$$

4) continuity of pressure and normal component of velocity in the direction vertical to the wake

$$\left.\begin{aligned}\Delta p &= 0\\ \nabla\varphi\cdot n &= 0\end{aligned}\right\} \quad \text{for} \quad z_W=(x, y)\in L_W. \tag{5}$$

The boundary conditions are complemented with an initial condition which states that for time $t<0$

$$L_W = 0 \quad \text{and} \quad U\cdot n = \text{const} \tag{6}$$

i.e. the profile started an unsteady motion at time $t=0$. The problem defined above reduces to the Neumann's outer problem for Laplace equation.

From the three basic solutions of two-dimensional Laplace equation (source, vortex and dipole), as it was mentioned above, the continuous distribution of vortices was chosen as the most suitable to unsteady flow about a profile with varying lift. The choice of surface vorticity leads to simple relation for lift and moment as a function of vorticity distribution on a profile and is satisfactorily accurate to calculate initial values of lift and moment as well as the shape of wake. A thin profile, as it was mentioned above, is replaced with a continuous distribution of surface vorticity on the segment $\langle -1, 1\rangle$ and the wake is replaced with an also continuous distribution along line $L_W$. Its shape is not known and depends on conditions of flow about the profile. As Laplace equation is a linear one, each superposition of basic solutions fulfils it. This statement applies also to the flow fields generated by a continuous distribution of surface vorticity along lines $L_P$ and $L_W$, see Fig. 1. Owing to the condition of impermeability of a thin profile we reach the following equation of vortex distribution on surface along lines $L_P$ and $L_W$:

$$\frac{1}{2\pi}\int\limits_{L_P}\frac{\gamma_P(x, t)}{x-\xi}\,dx+\frac{1}{2\pi}\int\limits_{L_W}\frac{\gamma_W(x, t)}{z_P-z_W}\,dx+v(\xi, t)=0, \tag{7}$$

where $z_P$, $x$ and $\xi\in L_P$, $z_W\in L_W$, and $t$ denotes time.

The first component in Eq. 7 describes the velocity component normal to the profile by the continuous distribution of surface vorticity on the profile, the second component of the equation describes the velocity component normal to the profile and induced by free vortices in the wake. The third component of the equation describes also the velocity component normal to the profile. This component can be due to some external reasons, for example the existence of other profiles or free vortices whose origin is not always connected with the profile under consideration. Of course, in the case of a quasi-stationary flow the second term equals zero, whereas both the vorticity distribution on the profile surface $\gamma_P(x, t)$ and the function $v(\xi, t)$ remain time-dependent. In the case of a stationary flow the dependence on time disappears and Eq. 7 reduces to CSIE, that is

$$\frac{1}{2\pi}\int \frac{\gamma_P(x, t)}{x-\xi}\,dx = -v(\xi). \tag{8}$$

The cases of stationary flow about a flat plate will be used for checking and comparing the accuracy of Belotserkovski method and that of Gorelov and Kulyaev with Gauss- and Radau-Jacobi quadratures as applied to solution of a CSIE.

Dynamic characteristics of profile motion (pressure distribution, lift and moment acting on a profile) are received from the Cauchy-Lagrange integral

$$p = w^2 - w_0^2 - 2\frac{\partial\varphi}{\partial t}, \tag{9}$$

where $w$ – liquid particles motion velocity, $w_0$ – relative velocity of liquid particles motion.

Cauchy-Lagrange integral if applied to the bottom and upper side of a profile gives nondimensional pressure differences

$$\Delta p = p_+ - p_- = w_{0+}^2 - w_{0-}^2 - 2\frac{\partial}{\partial t}(\varphi_- - \varphi_+) \tag{10}$$

and after suitable conversions

$$\Delta p = 2\left[w_{0t}\gamma_P - \frac{\partial\Gamma}{\partial t}\right], \tag{11}$$

where

$$w_{0t} = \frac{w_{0+} + w_{0-}}{2},$$

$$\gamma_P = \frac{w_{0+} - w_{0-}}{2}$$

and

$$\Gamma = \varphi_- - \varphi_+ .$$

The lift coefficient $C_L$ and the moment coefficient $C_M$ are calculated from the following formulae

$$C_L = \int_{L_P} p(x)\,n\,dx, \tag{12a}$$

$$C_M = \int_{L_P} p(x)\,x\,dx. \tag{12b}$$

The nonlinear problem of unsteady flow about a thin profile by inviscid and incompressible fluid reduces to finding the solution of Eq. 7 and particularly to finding the intensity of surface vorticity $\gamma_P(x, t)$ on the profile, the value of intensity of free surface vorticity $\gamma_W(\zeta, \eta, t)$ and its shape, that is to finding the points $z_W = (\zeta, \eta)$. The solution is searched for by means of the discrete vortex method, i.e. the method of discretization of surface vorticity intensity and discretization of time [8, 9]. For each time step $j$, $N$ values of surface vorticity intensity on the profile $\gamma_P(i, j)$ and the value of free surface vorticity intensity $\gamma_W(1, j)$ at the first point behind the profile is searched. The remaining $j-1$ values of free surface vorticity intensity were already determined in preceding time steps. In the following time steps $j-1$ new positions of vortices which form free surface vorticity are calculated. The position of the first vortex in the free surface vorticity is a free parameter of the whole method of solution. In this way the problem is reduced to that of solving the set of $N$ equations with $N+1$ unknowns, as one searches for $N+1$ values of intensities of surface vorticity at points $x_i$ and $\zeta_1$, and the boundary condition is checked in $N$ control points lying on the profile. The lacking $N+1$ equation is found from the condition of conservation of the whole system circulation. The most important factor determining the accuracy of solution of the problem of flow about a single profile is the accuracy of determining the intensity of the surface vorticity $\gamma_P(i, j)$ and $\gamma_W(1, j)$. The accuracy of $\gamma_P(i, j)$ and $\gamma_W(1, j)$ evaluation affects indirectly the accuracy determination of the free surface vorticity shape, $(\zeta, \eta)$. In the next time step the accuracy of $\gamma_P(i, j)$ and $\gamma_W(1, j)$ calculation is also affected. It is because the errors of the surface intensities evaluation add to each other in the following steps, that's why the accuracy of $\gamma_P(i, j)$ and $\gamma_W(1, j)$ determination is so important in the whole method. Hence, it is advantageous to consider the accuracy of CSIE (8) solution in different methods as dependent on the number $N$ of points $x_1$ and $\xi_k$ and their position on the profile.

## 3. CSIE Solution by Means of the Method of Belotserkovski and the Method of Gorelov and Kulyaev

As it was mentioned at the beginning, the method of Belotserkovski as well as the method of Gorelov belong to direct methods of CSIE solution. Their main idea consists in dividing the $\langle -1, 1 \rangle$ segment into $N$ equal parts of $2/N$ length and finding the positions $x_i$ and $\xi_k$ of discrete vortices and control points, respectively. Use is being made of the following formulae

$$\left.\begin{aligned} x_i &= l_1 + (i + \mu_i) h \\ \xi_k &= l_1 + (k + \nu_k) h \end{aligned}\right\} i,\ k = 1, 2, 3, \ldots, N,\ h = 2/N. \tag{13}$$

In case of the method of Belotserkovski the coefficients $\mu_i$ and $\nu_k$ are constant and equal $\mu = 0.25$ and $\nu = 0.75$, respectively. In case of the method of Gorelov and Kulyaev the coefficients are variable and take different values in the leading and trailing edge vicinity as well as in the middle region. Moreover, these coefficients take different values in case when one needs precise calculation of surface vorticity intensity on a profile [2], that is to calculate the values of the integrand, and slightly different values when the precise calculation of the value of the integral is needed [4].

For such a set of vortices and control points the integral equation of Cauchy type reduces to a set of $N$ linear equations

$$\sum_{i=1}^{N} \frac{\gamma_P(x_i)}{x_i - \xi_k} = -2\pi v(\xi_k). \tag{14}$$

The accuracy and convergence of the CSIE solution by means of the Belotserkovski method was examined and shown in [3]. As it was mentioned above the accuracy of this method of surface vorticity intensity determination is particularly small for the points close to the leading and trailing edge and practically it does not increase with rising $N$ [2]. According to [2] for a stationary flow about a flat plate for $N \to \infty$ the relative error of determination of the surface vorticity intensity equals $-0.112$, and the error close to the leading edge equals 0.329.

In order to reduce the error of determination of the surface vorticity intensity in the vicinity of the edge of a profile D. N. Gorelov and R. L. Kulyaev [2] propose nonuniform distribution of points of vortices displacement and control points. However, they remain the division of a profile into $N$ equal segments. According to [2] the accuracy of determination of the value of surface vorticity increases significantly. For the presented case of a stationary flow about a flat plate the relative error of the surface vorticity determination, assessed basing on an analytical solution, lies for the whole length of the profile within the range of 1 – 2%. In [2] D. N. Gorelov and R. L. Kulyaev present the values of coefficients $\mu_i$ and $v_k$ for $N = 15 \div 20$ with the accuracy of two positions after the decimal point with the simultaneous restriction that the more precise values of the coefficients $\mu_i$ and $v_k$ should be used to $N > 20$.

## 4. CSIE Solution by Means of the Method of Jacobi Quadratures

Jacobi quadratures (Mehler) have been used for many years in calculations of regular integrals [18]. However, only recently one managed to apply them to calculate singular integrals and in particular to calculate integrals of Cauchy type [15 ÷ 17]. The basic difference that distinguishes the method of Jacobi quadratures from the method of Belotserkovski and the method of Gorelov is a nonuniform distribution of points of vortices displacement and control points on a profile, and in case of Jacobi quadratures such a non-uniform distribution of points is a typical feature of the method. A set of most important formulae and definitions referring to Jacobi quadratures and their application to solution of CSIE will be presented now.

The basic feature of Jacobi quadratures is the fact that they are associated with the weight function $w(x)$

$$w(x) = (1-x)^{\alpha}\,(1-x)^{\beta}, \tag{15}$$

where the coefficients $\alpha$ and $\beta$ are greater than $-1$. For some special values of the $\alpha$ and $\beta$ coefficients the Jacobi quadratures reduce to some other well known quadratures – in this way for $\alpha = \beta = 0$ we obtain the Legendre quadrature, and for

$\alpha=\beta=\pm 1/2$ the Chebyshev quadrature is derived. In the case of interest, $\alpha=-\beta=\pm 1/2$, this is the Gauss-Jacobi quadrature. The quadratures mentioned above belong to the so-called open quadratures or the quadratures without pre-assigned abscissae. Another type of quadratures are the quadratures in which one or two abscissae are pre-assigned, i.e. pre-determined. Most often this refers to one or two ends of the integration region. In case of choosing one ending point as a pre-assigned abscissa, the quadrature is called a semi-closed one or Radau quadrature, and in case of both ending points being chosen as pre- assigned abscissae a closed quadrature or Lobatto quadrature is obtained. In our considerations we shall consider only the open Gauss-Jacobi quadrature and the semi-closed Radau-Jacobi quadrature, whereas the pre-assigned abscissa coinciding with the trailing edge of a profile, that is point $+1$. It is also possible to consider Radau-Jacobi quadrature with a pre-assigned abscissa $-1$, but in case of fluid mechanics it does not seem to be so useful as the Lobatto-Jacobi quadrature. Radau-Jacobi quadrature with abscissas $+1$ can be valid in fluid mechanics in so far as the fulfilment of Kutta-Joukovski condition about flow in the edge is assumed as usual in case of flow about a profile. This condition can be reduced to pre-assignment of zero value of intensity of surface vorticity at the trailing edge. In case of the weight function being used the integrand is

$$\gamma_P(x)=w(x)\cdot g(x). \tag{16}$$

The formulae of Gauss-Jacobi quadrature for singular integrals of Cauchy type are as follows [6]

$$\int_{-1}^{1}\frac{w(x)\cdot g(x)}{x-\xi}\,dx=\sum_{i=1}^{N}A_i\frac{g(x_i)}{x_i-\xi}-\frac{\Pi_N^{(\alpha,\beta)}(\xi)}{P_N^{(\alpha,\beta)}(\xi)}f(\xi)+E_N,\quad x_i\neq\xi, \tag{17}$$

where $A_i$ and $x_i$ are the weight coefficients and abscissae of Gauss-Jacobi quadrature respectively and are given with the following formulae

$$A_i=\frac{2\pi(1-x_i)}{2N+1}, \tag{18}$$

$$x_i=\cos\left(\frac{2\pi i}{2N+1}\right) \tag{19}$$

and $N$ – the number of abscissae. The abscissae $x_i$ are the roots of the Jacobi polynominals $P_N^{(\alpha,\beta)}(x)$ with the same parameters as the weight function $w(x)$ and $E_N$ is the approximation error. The approximation is accurate for the functions $f(x)$ being the polynomials of degree $2N$ [6]. In case of integral equation which describes the flow about thin profile it is possible to prove that $\alpha=1/2$ and $\beta=-1/2$, and the CSIE index $\nu$ is equal zero. And continuing after [6] the equation for $\Pi_N^{(\alpha,\beta)}(x)$ takes the form

$$\Pi_N^{(\alpha,\beta)}(x)=\Gamma(\alpha)\,\Gamma(\beta)\,P_{N-\nu}^{(-\alpha,-\beta)}(x), \tag{20}$$

where $\Gamma$ – gamma function.

If a condition is formulated that the control points are chosen as the roots of the Jacobi polynomial $P_N^{(-\alpha,-\beta)}(x)$, then the Cauchy integral can be reduced (if $E_N$ is neglected) to the following sum

$$\int_{-1}^{1} \frac{w(x)\,g(x)}{x-\xi}\,dx = \sum_{i=1}^{N} A_i \frac{g(x_i)}{x_i-\xi_k} \tag{21}$$

and CSIE is reduced to a set of $N$ linear equations,

$$\sum_{i=1}^{N} A_i \frac{g(x_i)}{x_i-\xi_k} = -v(\xi_k). \tag{22}$$

It is possible to prove that the roots $\xi_k$ of the Jacobi polynomials $P_{N-\nu}^{(-\alpha,-\beta)}(x)$ have the sign opposite to the roots $x_i$ of Jacobi polynomial $P_N^{(\alpha,\beta)}(x)$ [11]. In this way we have finished a brief presentation of Gauss-Jacobi quadrature for CSIE. It may be advantageous to remind that in case of Gauss-Jacobi quadrature the points $x_i$ and $\xi_k$ are determined very easily as it can be done from Eq. 19.

In the following section we shall consider Radau-Jacobi quadrature briefly for the case of one pre-assigned abscissa, i.e. $x_N = 1$. Radau-Jacobi quadrature reduces CSIE to a set of $N$–1 linear equations (22), but the abscissae $x_i$ and the control points $\xi_k$ are the roots of the following Jacobi polynomials:

$$(x-1)\,P_N^{(3/2,\,1/2)}(x) = 0, \tag{23}$$

$$P_N^{(-3/2,\,1/2)}(x) = 0, \tag{24}$$

and the formulae for weight coefficients $A_i$ are different [10], that is

$$A_i = \frac{2^{\alpha+\beta}\,\Gamma(N+\alpha)\,\Gamma(N+\beta)}{(N+1)!\,(N+\alpha)\,\Gamma(N+\alpha+\beta+1)} \cdot \frac{1+x_i}{[P_{N-1}^{(1/2,\,1/2)}(x_i)]^2}, \tag{25}$$

where $P_{N-1}^{(\alpha,\beta)}(x)$ is also the Jacobi polynomial of $N$–1 degree. However, as it can be proved [11], only $N$–1 roots of the Jacobi polynomial, Eq. (23), lie within the $(-1, 1)$ range. Hence, we shall have finally only $N-1$ linear equations. That is why the $N$-th equation must be searched for from another physical condition. In [7] N. I. Ioakimidis proposed a technique of determination of the lacking equation. This technique will be discussed below. In case of CSIE of the first kind it reduces to solving the following equation:

$$\sum_{i=1}^{N} A_i\, g(x_i) = F, \tag{26}$$

where $F$ is constant and equals

$$F = \int_{-1}^{1} w^{-1}(x)\, g(x)\, dx. \tag{27}$$

Integration is performed here with the aid of Radau-Jacobi quadrature.

Table 1

**Values of weight coefficients $A_i$, abscissae $x_i$ and control points $\xi_k$ for the Radau-Jacobi quadrature at $N=20$**

| | $A_i$ | $x_i$ | $\xi_i$ |
|---|---|---|---|
| 1 | .31387091E 00 | −.99691348 | |
| | | | −.98767294 |
| 2 | .31001018E 00 | −.97233535 | |
| | | | −.95099528 |
| 3 | .30238380E 00 | −.92378429 | |
| | | | −.89087015 |
| 4 | .29117955E 00 | −.85245579 | |
| | | | −.80877804 |
| 5 | .27667331E 00 | −.76010619 | |
| | | | −.70674031 |
| 6 | .25922227E 00 | −.64900944 | |
| | | | −.58726949 |
| 7 | .23925614E 00 | −.52190111 | |
| | | | −.45330732 |
| 8 | .21726655E 00 | −.38191103 | |
| | | | −.30815241 |
| 9 | .19379495E 00 | −.23248621 | |
| | | | −.15537894 |
| 10 | .16941930E 00 | −.77305986E-01 | |
| | | | .12513035E-02 |
| 11 | .14473980E 00 | .79808599E-01 | |
| | | | .15788157 |
| 12 | .12036414E 00 | .23498888 | |
| | | | .31065513 |
| 13 | .96892524E-01 | .38441381 | |
| | | | .45581020 |
| 14 | .74902900E-01 | .52440411 | |
| | | | .58977265 |
| 15 | .54936712E-01 | .65151281 | |
| | | | .70924397 |
| 16 | .37485569E-01 | .76261024 | |
| | | | .81128264 |
| 17 | .22979113E-01 | .85496121 | |
| | | | .89337684 |
| 18 | .11774324E-01 | .92629311 | |
| | | | .95350809 |
| 19 | .41459084E-02 | .97485713 | |
| | | | .99022319 |
| 20 | .29470850E-03 | 1.00000000 | |

Another problem is caused by determination of the roots of Jacobi polynomials, Eqs. (23) and (24), and particularly their accuracy. In order to determine the roots of Jacobi polynomials the method presented in [12] for orthogonal polynomials and the finite procedures for calculation of eigenvalues of tridiagonal matrix is used. In Table 1 the values $x_i$, $\xi_k$ and $A_i$ for Radau-Jacobi quadrature for $\alpha=1/2$ and $\beta=-1/2$ and $N=20$ are presented.

## 5. Comparison of the method of Belotserkovski and the method of Jacobi quadratures

As it was mentioned before both methods were checked for two cases of flow about flat plate, i.e. the stationary flow about flat plate and the flow about flat plate with stationary vortex in the wake. All the comparing calculations were performed for the number of abscissae equal to $N=20$. Relative error of the method is determined according to the formula

$$\Delta\gamma(x_i)=\frac{\gamma_P(x_i)-\gamma_a(x_i)}{\gamma_a(x_i)}, \tag{28}$$

where $\gamma_P(x_i)$ and $\gamma_a(x_i)$ – the values of the surface vorticity intensity at a point $x$ previously determined by numerical method and the known analytical solution.

The calculations by the method of Belotserkovski and the method of Gorelov and Kulyaev were performed for the stationary flow about a flat plate. In this case, as it is known, $v(x)=\sin\,\delta$, where $\delta$ is the angle of inclination of a profile to the oncoming velocity. In the case of Belotserkovski method the results were the same as in [3], i.e. for $i=1$ the error of determination of intensity of surface vorticity equals $\Delta\gamma(1)=-0.113$ and for $i=-20$ the error $\Delta\gamma(20)=0.023$, in all of the remaining points the error was below $\Delta\gamma(i)=0.01$. Unfortunately it proved to be impossible to reach the same accuracy of calculations for the method of Gorelov and Kulyaev as that obtained in [2], that is, the error below $1\div2\%$ for all of abscissae. The results received are as follows: for $i=1$ the error $\Delta\gamma(1)=0.153$, for $i=19$ the error $\Delta\gamma(19)=-0.034$ and for $i=20$ the relative error of intensity $\Delta\gamma(20)=0.144$. At the remaining 17 points the error does not overcome $1\div2\%$, in fact. One may suppose that the authors [2] used the more precise values of $x_i$ and $\xi_k$ then it was presented. Such a possibility of increasing the accuracy of calculation of the surface vorticity is presented in the paper by the authors themselves [2].

Very accurate results were obtained in case of Jacobi quadratures. For the Gauss-Jacobi quadrature the error of surface vorticity determination does not exceed $10^{-13}$ for all $i$ values. Thus the value of surface vorticity was determined with accuracy practically equal to the accuracy of the computer calculations. A little less accuracy was obtained in calculation of the surface vorticity value in case of Radau-Jacobi quadratures method. However, also in this case it was very good, as the error of determination of surface vorticity for all of the abscissae was smaller then $10^{-9}$. This accuracy is better than that of the Belotserkovski or Gorelov and Kulyaev methods. The difference in the accuracy of both methods of Jacobi quadratures results, as one may suppose from the accuracy of determination of $x_i$ and $\xi_k$ points, because in the method of Gauss-Jacobi quadratures they are given in simple formulae, whereas in the method of Radau-Jacobi quadrature the points are determined with the aid of special programs and additional procedures.

Let's discuss the results of comparison of the flow about a flat plate with stationary vortex in the wake. In fact this is the basic flow pattern used by Belotserkovski and Kulyaev [2] for solution of the nonlinear problems of unsteady flow about thin profile by means of the discrete vortex method. In both papers the authors use $N$ vortices on a thin profile and one vortex in the wake. Its intensity is unknown, but its position is determined and placed behind the trailing edge. Whereas in [2] a rule for placing this first free vortex in the wake is given, in [1] no such rule is given. That is why the examination of the above methods of CSIE solution for several positions of vortices $\zeta_1 = 1.02$, 1.03, 1.05, 1.1 and 1.15 was performed.

Suitable formulae (here only cited) for the distribution of surface vorticity on a thin profile and the total circulation around a profile can be found in [13]. The distribution of the surface vorticity is given according to [13] by the following formula

$$\gamma_a(x) = \frac{1}{2\pi}\frac{\Gamma'}{\zeta_1 - x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\zeta_1+1}{\zeta_1-1}}, \tag{29}$$

where $\Gamma'$ – intensity of the vortex, $\zeta_1$ – distance from the vortex to the middle of the profile.

The total circulation of velocity around a profile is given by the following formula

$$\left(\frac{\Gamma}{\Gamma'}\right)_a = \sqrt{\frac{\zeta_1+1}{\zeta_1-1}} - 1.$$

In order to determine the relative errors of the surface vorticity intensities calculation and those of the calculation of velocity circulation around a profile with the aid of different methods, Eq. (28) was used. In Fig. 2 the accuracy of calculation by means of Belotserkovski method of surface vorticity on thin profile with stationary vortex at the distance $\zeta_1 = 1.02$, 1.03, 1.05, 1.1 and 1.15 from the profile centre, i.e. at the distance 0.02, 0.03, 0.05, 0.10 and 0.15 behind the trailing edge is presented. In Figs. 3÷5 the accuracies of surface vorticity calculation for the

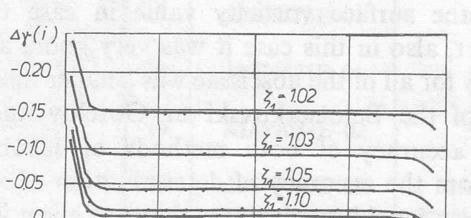

Fig. 2. Relative error $\Delta\gamma(i)$ of the surface vorticity intensity determination in the method of Belotserkovski for different distances of free vortex in the wake of a profile; $\zeta_1 = 1.02$, 1.03, 1.05, 1.10, 1.15

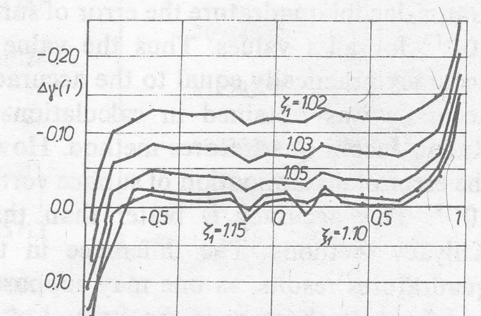

Fig. 3. Relative error $\Delta\gamma(i)$ of the surface vorticity intensity determination in the method of Gorelov and Kulyaev for different distances of free vortex in the wake of a profile; $\zeta_1 = 1.02$, 1.03, 1.05, 1.10, 1.15

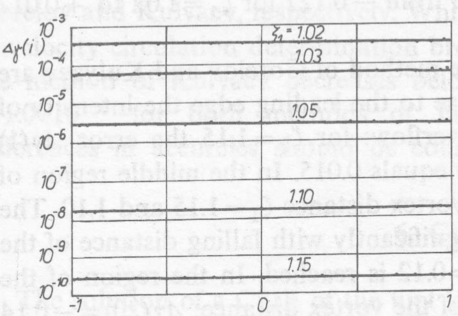


Fig. 4. Relative error $\Delta\gamma(i)$ of the surface vorticity intensity determination in the method of Gauss-Jacobi quadratures for different distances of free vortex in the wake of a profile; $\zeta_1 = 1.02$, 1.03, 1.05, 1.10, 1.15

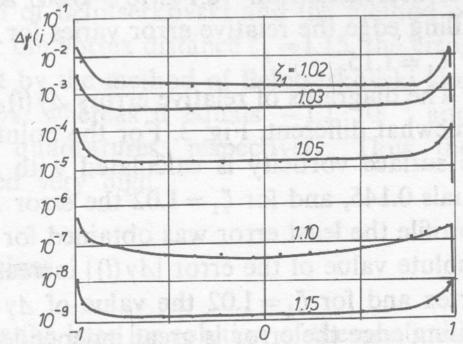


Fig. 5. Relative error $\Delta\gamma(i)$ of the surface vorticity intensity determination in the method of Radau-Jacobi quadratures for different distances of free vortex in the wake of a profile; $\zeta_1 = 1.02$, 1.03, 1.05, 1.10, 1.15

parameters as in Fig. 2 are shown. They refer to the method of Gorelov and Kulyaev, the method of Gauss-Jacobi and the method of Radau-Jacobi quadrature, respectively. Table 2 presents the accuracy of the calculation of velocity circulation around a thin profile calculated with the four methods mentioned above and the distance of vortices as in Figs. 2÷5. From the analysis of Figs. 2÷5 it is clear that the determination of the accuracy values $\Delta\gamma(i)$ is highly dependent on the position of the $\zeta_1$ point. The greatest relative errors in determination of $\Delta\gamma(i)$ occur in case of $\zeta_1 = 1.02$, and the smallest for $\zeta_1 = 1.15$. This rule is being observed in all discussed methods.

Table 2

Relative error $\Delta(\Gamma/\Gamma')$ of the velocity circulation determination around a profile for different distances of free vortex in the wake of a profile ($\zeta_1 = 1.02$, 1.03, 1.05, 1.10, 1.15) and for different calculation methods: B – method of Belotserkovski, G & K – Gorelov and Kulyaev method, G & J – Gauss-Jacobi quadratures method, R & J – Radau-Jacobi quadratures method

| | $\Delta(\Gamma/\Gamma')$ | | | |
|---|---|---|---|---|
| $\zeta_1$ | G & J | R & J | B | G & K |
| 1.02 | −6.2E-04 | −7.5E-04 | −0.163 | −0.132 |
| 1.03 | −1.0E-04 | −4.2E-04 | −0.103 | −0.082 |
| 1.05 | −5.9E-06 | 8.1E-06 | −0.052 | −0.041 |
| 1.10 | −1.4E-07 | −7.8E-07 | −0.018 | −0.014 |
| 1.15 | −1.1E-07 | −1.0E-09 | −0.009 | −0.007 |

In case of Belotserkovski method the error of the intensity $\Delta\gamma(i)$ determination is not constant for all $i$, the greatest value is reached for $i=1$. For $\zeta_1 = 1.02$ the error $\Delta\gamma(1)$ is equal $-0.24$, and for $\zeta_1(1) = -0.12$. In the middle region of a profile the

$\Delta(\Gamma/\Gamma') = -0.163$ and $-0.132$ in the method of Belotserkovski and the method of Gorelov and Kulyaev, respectively. While for the vortex distance $\zeta_1 = 1.15$, the error of velocity circulation determination brought by the method of Belotserkovski and the method of Kulyaev decreases below 1%, whereas it equals $-1.1 \cdot 10^{-7}$ and $-1.0 \cdot 10^{-9}$ for both methods of Jacobi quadratures, respectively. Thus the differences in accurcies should be considered very high.

## 6. Conclusions

The solution of a CSIE of the first kind is the base for solution of the nonlinear problems of unsteady flow about a thin profile by an inviscid and incompressible fluid. The methods of solution used hitherto brought significant numerical errors. The aim of this paper is to propose the methods of Jacobi quadratures for solution of CSIE. The reason is accuracy of these methods being by several orders of magnitude better than that of the method of Belotserkovski and its modification – the method of Gorelov and Kulyaev, both of them having been used until now. In order to compare different methods of CSIE solution the known analytical solutions of stationary flow about a flat plate and a flat plate with a vortex in the wake have been applied. The second example is particularly important as it is the basis of the method of discrete vortices, generally applied for solution of the nonlinear problems of unsteady flow about thin profile by a perfect fluid. Due to the simplicity of abscissae $x_i$ and control points $\zeta_k$ determination, the Gauss-Jacobi quadrature is easier to use than the Radau-Jacobi quadrature. The application of the Radau-Jacobi quadrature gives a little worse accuracy of surface vorticity intensity determination for the cases under consideration than the Gauss-Jacobi one because of the greater error of abscissae and control points determination. The Radau-Jacobi quadrature is a semi-closed one which has the advantage of one of the abscissae coinciding with the trailing edge. That feature may be of some significane in fluid mechanics.

Received by the Editor, June 1987.

### References

[1] S. M Belotserkovski, M. I. Nisht, *Otryvnoe i bezotryvnoe obtekanie tonkikh kryl'ev idealnoi zhidkost'yu.* Moskva 1978.

[2] D. N. Gorelov, R. L. Kulyaev, *Nonlinear problem of unsteady flow of an incompressible fluid past a slender profile.* Mekh. Zhidk. i Gaza, No 6, 1971, pp. 38-48.

[3] I. K. Lifanov, Ya. E. Polon'ski, *Obosnovanie chislennogo metoda "diskretnykh vikhrei" resheniya singulyarnykh integralnykh uravnenii.* PMM, 1975, pp. 742-746.

[4] D. N. Gorelov, *Lokal'naya aproksimatsiya vikhregogo sloya sistemoi diskretnykh vikhrei.* PMTF, No. 5, 1980, pp. 76-82.

[5] Z. Wierciński, *Comparison of the method of Belotserkovski and the method of Jacobi quadratures applied to the discrete vortex method in nonlinear problems of unsteady fluid mechanics.* Int. Sem. Engng. Appl. of Surface and Cloud Vorticity Method, Wrocław-Szklarska Poręba, Poland, April 15-18, 1986.

[6] N. I. Ioakimidis, P. S. Theocaris, *On the numerical solution of a class of singular integral equations.* J. Math. Phys. Sci., 1977, Vol. 11, No. 3, pp. 219-235.

[7] N. I. Ioakimidis, *A remark on the application of closed and semiclosed quadratures rules to the direct numerical solution of singular integral equations.* J. Comp. Physics, 1981, Vol. 42, pp. 396-402.

[8] S. M. Belotserkovski, I. K. Lifanov, *Chislennye metody v singulyarnykh integral'nykh uravneniyakh.* „Nauka", Moskva 1985.

[9] M. M. Chawla, T. R. Ramakrishnan, *Modified Gauss-Jacobi quadrature formulas for the numerical evaluation of Cauchy type singular integrals.* BIT 1974, Vol. 14, pp. 14-21.

[10] Z. Kopal, *Numerical analysis.* Chapman & Hall Ltd., London 1961.

[11] G. Szegö, *Orthogonal polynomials.* American Math. Society, New York 1959.

[12] J. Stoer, R. Bulirsch, *Introduction to numerical analysis.* Springer Verlag 1976.

[13] Th. v. Karman, W. R. Sears, *Airfoil theory for non-uniform motion.* J. Aerospace Sci. 5, 1938, pp. 379-390.

## Porównanie metody Biełocerkowskiego i metody kwadratur Jakobiego w zastosowaniu do rozwiązania nieliniowych problemów nieustalonej mechaniki płynów metodą dyskretnych wirów

Streszczenie

W pracy porównano dokładność zastosowania metody Biełocerkowskiego i metody kwadratur Jacobiego w zastosowaniu do rozwiązania nieliniowych problemów nieustalonej mechaniki płynów. Stwierdzono, że dokładność metody kwadratur Jacobiego do obliczeń rozkładu wirowości na cienkim profilu metodą dyskretnych wirów jest o kilka rzędów lepsza niż dotychczas stosowanej metody Biełocerkowskiego. Zasadnicza różnica odróżniająca obie metody polega na zastosowaniu nierównomiernego rozkładu punktów położenia wirów i punktów kontrolnych na profilu (kwadratura Jacobiego) w miejsce rozkładu równomiernego (metoda Biełocerkowskiego).

## Сравнение метода Белоцерковского и метода квадратур Якоби в применении к решениям нелинейных проблем нестационарной механики жидкостей методом дискретных вихрей

Резюме

В работе сравнена точность применения метода Белоцерковского и метода квадратур Якоби в применении к решениям нелинейных проблем нестационарной механики жидкостей. Доказывается, что точность метода квадратур Якоби для расчётов распределения завихрённости на тонком профиле методом дискретных вихрей на несколько порядков лучше чем точность применяемного до сих пор метода Белоцерковского. Основная разница отличающая оба методы заключается в применении неравномерного распределения точек расположения вихрей и контрольных пунктов на профиле (квадратура Якоби), вместо равномерного распределения (метод Белоцерковского).