POLSKA AKADEMIA NAUK
INSTYTUT MASZYN PRZEPŁYWOWYCH

PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH

TRANSACTIONS OF THE INSTITUTE OF FLUID-FLOW MACHINERY

86

WARSZAWA – POZNAŃ 1983
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH

poświęcone są publikacjom naukowym z zakresu teorii i badań doświadczalnych w dziedzinie mechaniki i termodynamiki przepływów, ze szczególnym uwzględnieniem problematyki maszyn przepływowych

*

THE TRANSACTIONS OF THE INSTITUTE OF FLUID-FLOW MACHINERY

exist for the publication of theoretical and experimental investigations of all aspects of the mechanics and thermodynamics of fluid-flow with special reference to fluid-flow machinery

REDAKCJA – EDITORIAL OFFICE

Instytut Maszyn Przepływowych PAN
ul. Gen. Józefa Fiszera 14, 80-952 Gdańsk, skr. pocztowa 621, tel. 41-12-71

ISBN 83-01-04798-4
ISSN 0079-3205

PAŃSTWOWE WYDAWNICTWO NAUKOWE – ODDZIAŁ W POZNANIU

Nakład 300+90 egz.	Oddano do składania 8 XI 1982 r.
Ark. wyd. 10. Ark. druk. 8,5	Podpisano do druku 31 V 1983 r.
Pap. druk. sat. kl. V, 70 g	Druk ukończono w czerwcu 1983 r.
Nr zam. 47/203	E-9/324. Cena zł 100,–

DRUKARNIA UNIWERSYTETU IM. A. MICKIEWICZA W POZNANIU

PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH
1983 Zeszyt 86

JURAND RYTERSKI

Gdańsk*

Solutions of Certain Plane Problems of Stationary Vibrations of Elastic Body

In this paper we consider the plane problem of stationary vibrations of an isotropic homogeneous elastic body. We take into account solutions in the sense of distribution theory. In the introduction we explain the idea of a vector distribution and using the theory we find the equivalents of Green formula.

The solutio ı method of these problems is based on the idea presented in paper [2]. It allowed us to obtain the closed form solutions of some specific problems in the unbounded domain.

The similar problems were considered in papers [3] and [4] where other methods were applied.

1. Some remarks about vector distributions

Let $\mathscr{D}=\mathscr{D}_1$ be the space of test functions, and $\mathscr{D}'=\mathscr{D}'_1$ be the distribution space. Denote $\mathscr{D}_3=\mathscr{D}\times\mathscr{D}\times\mathscr{D}$ and $\mathscr{D}'_3=\mathscr{D}'\times\mathscr{D}'\times\mathscr{D}'$ the test vector space and the vector distribution space, respectively. When u is a vector distribution i.e. $u\in\mathscr{D}'_3$ then we can represent it as:

$$u=i_1 u_1+i_2 u_2+i_3 u_3,$$

where i_1, i_2, i_3 are versors of the coordinate system, and $u_p\in\mathscr{D}'$ $(p=1, 2, 3)$.

The value of the vector distribution u for the test vector $\varphi=i_1\varphi_1+i_2\varphi_2+i_3\varphi_3$ is defined by the equality:

$$(u, \varphi)=(u_1, \varphi_1)+(u_2, \varphi_2)+(u_3, \varphi_3)$$

$$\forall\, \varphi\in\mathscr{D}_3 \Leftrightarrow \forall\, \varphi_p\in\mathscr{D} \quad (p=1, 2, 3)$$

Let us proceed to the definition of fundamental differential operations

1. Let $f\in\mathscr{D}'$, grad f is defined then as a vector distribution, and the following equality is valid:

$$(\operatorname{grad} f, \varphi)=-(f, \operatorname{div}\varphi) \quad \forall\varphi\in\mathscr{D}_3 \tag{A}$$

2. If $u\in\mathscr{D}'_3$ then:

$$\operatorname{div} u\in\mathscr{D}'$$

---

* Technical University of Gdańsk.

$$(\operatorname{div} u, \varphi) = -(u, \operatorname{grad} \varphi) \quad \forall \varphi \in \mathscr{D}_1 \tag{B}$$

$$\operatorname{rot} u \in \mathscr{D}'_3$$

$$(\operatorname{rot} u, \varphi) = (u, \operatorname{rot} \varphi) \quad \forall \varphi \in \mathscr{D}_3 \tag{C}$$

In the plane case the vectors have obviously two components, and definition (A), (B) and (C) are valid.

2. Fundamental formulas for the plane case

We deal with two types of plane problems in the theory of elasticity – the plane strain problem and the plane stress problem. The equations of motion in both these cases have the same mathematical form. For the plane state of strain the motion equations are as follows:

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} = \rho \frac{\partial^2 v_1}{\partial t^2}, \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} = \rho \frac{\partial^2 v_2}{\partial t^2}, \tag{1}$$

where $\tau_{ik} (i, k=1, 2)$ denotes the component of the stress tensor, $v_i (i=1, 2)$ – the component of the displacement vector, ρ – density, $x_i (i=1, 2)$ – the point coordinate, t – time.

There exist following relations between the stress tensor component and the displacement vector component:

$$\left.\begin{aligned} \tau_{11} &= (\lambda+2\mu)\frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial v_2}{\partial x_2}, \\ \tau_{12} &= \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right), \\ \tau_{22} &= \lambda \frac{\partial v_1}{\partial x_1} + (\lambda+2\mu)\frac{\partial v_2}{\partial x_2}. \end{aligned}\right\} \tag{2}$$

In the case of stationary harmonic vibrations it is assumed that:

$$\begin{aligned} v_1(x_1, x_2, t) &= u_1(x_1, x_2)e^{i\omega t} \\ v_2(x_1, x_2, t) &= u_2(x_1, x_2)e^{i\omega t} \end{aligned} \tag{3}$$

and respectively

$$\tau_{j,k}(x_1, x_2, t) = \sigma_{jk}(x_1, x_2)e^{i\omega t} \tag{4}$$

$(j, k=1, 2)$ where ω is the vibration frequency.

From (1), (2) and (3) we obtain the basic system of equations

$$Lu = (\lambda+\mu)\operatorname{grad}\theta + \mu\Delta u + \omega^2 u = 0, \tag{5}$$

where $u = (u_1, u_2)$ denotes the displacement vector,

$$\theta = \operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$$

and $\rho = 1$ is assumed for simplicity.

Each distribution $u=(u_1, u_2)$ which fulfills the condition:

$$(Lu, \varphi)=0 \quad \forall \varphi \in \mathscr{D}_2,$$

where $\mathscr{D}_2=\mathscr{D}\times\mathscr{D}$ indicates the test vector space, is called the solution of Eq. (5).

Using the formula

$$\operatorname{grad}\theta=\Delta u+\operatorname{rot} V$$

where $V=\operatorname{rot} u=i_3\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)$

and respectively

$$\operatorname{rot} V=i_1\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)+i_2\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right)$$

we can write equation (5) in the form:

$$(\lambda+\mu)\operatorname{rot} V+(\lambda+2\mu)\Delta u+\omega^2 u=0. \tag{6}$$

It is easy to see that likewise the three-dimensional case, the vector function V and the scalar function θ satisfy the equations:

$$\mu\Delta V+\omega^2 V=0, \tag{7}$$

$$(\lambda+2\mu)\Delta\theta+\omega^2\theta=0, \tag{8}$$

and since $V=i_3v_3=i_3\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)$, also $\mu\Delta v_3+\omega^2 v_3=0$.

In order to define the differential operations on the two-dimensional vectors we introduce the concept of a simple layer and a double layer.

Def. 1. Let L be the contour, and $\mu(x)$ be the piecewise continuous function on L. The distribution $\mu\delta_L$ defined by the rule

$$(\mu\delta_L, \varphi):=\oint_L \mu(x)\varphi(x)\,ds \quad \forall\varphi\in\mathscr{D}, \tag{9}$$

where $\mathscr{D}$ is the test function space, will be called the simple layer with the density μ on the contour L.

Remark 1. For $x\in L$ we have $\mu\delta_L=0$ in view of the inclusion

$$\operatorname{supp}[\mu\delta_L]\subset L.$$

Def. 2. The double layer with the density $v(x)$ on the contour L is the distribution $-\frac{\partial}{\partial n}(v\delta_L)$ defined by the rule

$$\left(-\frac{\partial}{\partial n}(v\delta_L), \varphi\right):=\oint_L v(x)\frac{\partial\varphi}{\partial n}\,ds \quad \forall\varphi\in\mathscr{D}, \tag{10}$$

where $v(x)$ is the piecewise continuous function defined on L.

Remark 2. For $x \notin L$ we have $-\frac{\partial}{\partial n}(\nu\delta_L)=0$ in view of the inclusion

$$\operatorname{supp}\left(-\frac{\partial}{\partial n}(\nu\delta_L)\right)\subset L.$$

Remark 3. Let n always denote an external normal versor.

If $f\in C^1(\bar{D})\cap C^1(\bar{D}')$ then by definition of the derivative of a distribution we obtain

$$\frac{\partial f}{\partial x_i}=\left\{\frac{\partial f}{\partial x_i}\right\}+[f]_L\cos(nx_i)\delta_L \quad (i=1,2), \tag{11}$$

where $\{\partial f/\partial x_i\}$ denotes the classical derivative in case of its existence while

$$[f]_L=\lim_{\substack{x'\to x\\ x'\in D'}} f(x')-\lim_{\substack{x'\to x\\ x'\in D}} f(x') \qquad \forall x\in L,$$

where D is the domain bounded by the contour L, and $D'=R^2\setminus\bar{D}$. Hence

$$\operatorname{grad} f=\{\operatorname{grad} f\}+[f]_L n\delta_L. \tag{12}$$

Assuming that $f\equiv 0$ for $x\in D'$ we have

$$\operatorname{grad} f=\{\operatorname{grad} f\}-fn\delta_L \tag{12a}$$

since the external boundary value of the function f is equal to zero.

In the same way we obtain

$$\operatorname{div} u=\{\operatorname{div} u\}+[u]_L n\delta_L, \tag{13}$$

$$\operatorname{rot} u=\{\operatorname{rot} u\}+(n\times[u]_L)\delta_L, \tag{14}$$

and respectively

$$\operatorname{div} u=\{\operatorname{div} u\}-(un)\delta_L, \tag{13a}$$

$$\operatorname{rot} u=\{\operatorname{rot} u\}-(n\times u)\delta_L \tag{14a}$$

in case when the external boundary value of the vector u is zero.

By differentiating formula (11) we have

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=\left\{\frac{\partial^2 f}{\partial x_i\partial x_j}\right\}+\frac{\partial}{\partial x_j}([f]_L\cos(nx_i)\delta_L)+\left[\left\{\frac{\partial f}{\partial x_i}\right\}\right]_L\cos(nx_j)\delta_L, \tag{15}$$

where $f\in C^2(\bar{D})\cap C^2(\bar{D}')$ is assumed.

We assume furthermore that $u\in C^3(\bar{D})\cap C^3(\bar{D}')$ and $u=0$ for $x=(x_1,x_2)\in D'$.

For the mentioned above vector u we have

$$\Delta u=\{\Delta u\}-\frac{\partial u}{\partial n}\delta_L-\frac{\partial}{\partial n}(u\delta_L). \tag{16}$$

Taking into account formulas (5), (12), (13) and (16) we obtain

$$Lu=\{Lu\}-\left\{\mu\frac{\partial u}{\partial n}+(\lambda+\mu)\theta n\right\}\delta_L-\mu\frac{\partial}{\partial n}(u\delta_L)-(\lambda+\mu)\operatorname{grad}[(un)\delta_L], \tag{17}$$

where the second component of the right side of formula (17) denotes the simple layer, the third component – the double layer, and the fourth component is the distribution defined by the rule

$$(\operatorname{grad}[(u\,n)\delta_L],\,\varphi) = -((u\,n)\delta_L,\,\operatorname{div}\varphi) = -\oint_L (u\,n)\operatorname{div}\varphi\,ds \quad \forall\,\varphi\in\mathscr{D}_2 .$$

Formula (17) is the equivalent of Green formula, since if follows from

$$(Lu\,,\,\varphi) = (u\,,\,L\varphi) = \int_D uL\varphi\,dx$$

that

$$\int_D (uL\varphi - \varphi Lu)\,dx = \oint_L \Big[\Big(\mu\frac{\partial\varphi}{\partial n} + (\lambda+\mu)\,n\operatorname{div}\varphi\Big)u - \Big(\mu\frac{\partial u}{\partial n} + (\lambda+\mu)\,\theta n\Big)\varphi\Big]\,ds \quad \forall\,\varphi\in\mathscr{D}_2 . \qquad (18)$$

Taking advantage of the equality

$$\int_D \{\operatorname{grad}\theta\}\,\varphi\,dx = \oint_L \{\theta\}\,n\varphi\,ds - \int_D \{\theta\}\operatorname{div}\varphi\,dx$$

formula (18) can be rewritten as

$$\int_D [((\lambda+\mu)\operatorname{grad}\operatorname{div}\varphi + \mu\Delta\varphi + \omega^2\varphi)\,u - (\mu\Delta u + \omega^2 u)\,\varphi]\,dx =$$

$$= \oint_L \Big[\Big(\mu\frac{\partial\varphi}{\partial n} + (\lambda+\mu)\,n\operatorname{div}\varphi\Big)u - \mu\frac{\partial u}{\partial n}\varphi\Big]\,ds - (\lambda+\mu)\int_D \theta\operatorname{div}\varphi\,dx . \qquad (19)$$

Let us denote, as in paper [2]

$$T\varphi = 2\mu\frac{\partial\varphi}{\partial n} + \lambda n\operatorname{div}\varphi + \mu(n\times\operatorname{rot}\varphi),$$

$$N\varphi = \frac{\partial\varphi}{\partial n} - n\operatorname{div}\varphi + n\times\operatorname{rot}\varphi .$$

It is possible to present equation (19) in the form:

$$\int_D [((\lambda+\mu)\operatorname{grad}\operatorname{div}\varphi + \mu\Delta\varphi + \omega^2\varphi)\,u - (\mu\Delta u + \omega^2 u)\,\varphi]\,dx =$$

$$= \oint_L \Big[(T\varphi - \mu N\varphi)\,u - \mu\frac{\partial u}{\partial n}\varphi\Big]\,ds - (\lambda+\mu)\int_D \theta\operatorname{div}\varphi\,dx =$$

$$= \oint_L \Big[(T\varphi)\,u - \mu\Big(Nu + \frac{\partial u}{\partial n}\Big)\varphi\Big]\,ds - (\lambda+\mu)\int_D \theta\operatorname{div}\varphi\,dx \qquad (20)$$

since

$$\oint_L uN\varphi\,ds = \oint_L \varphi Nu\,ds .$$

After replacing equation (5) by equation (6), we obtain

$$\int_D [((\lambda+\mu)\operatorname{rot}\operatorname{rot}\varphi+(\lambda+2\mu)\Delta\varphi+\omega^2\varphi)u-((\lambda+2\mu)\Delta u+\omega^2 u)\varphi]dx=$$

$$=\oint_L\left[(T\varphi+\lambda N\varphi)u-(\lambda+2\mu)\frac{\partial u}{\partial n}\varphi\right]ds+(\lambda+\mu)\int_D \operatorname{rot} u \operatorname{rot}\varphi\, dx. \qquad (21)$$

The distributive form of formula (21) is as follows

$$Au:=(\lambda+\mu)\operatorname{rot} V+(\lambda+2\mu)\Delta u+\omega^2 u=$$

$$=\{Au\}-\left\{(\lambda+\mu)(n\times V)+(\lambda+2\mu)\frac{\partial u}{\partial n}\right\}\delta_L-$$

$$-(\lambda+\mu)\operatorname{rot}[(n\times u)\delta_L]-(\lambda+2\mu)\frac{\partial}{\partial n}(u\delta_L). \qquad (22)$$

3. The fundamental solutions for the plane case

At the beginning we consider the equation

$$Au=-(\lambda+2\mu)\operatorname{grad}\delta(x-x_0) \quad \text{where } x=(x_1, x_2)$$

otherwise

$$(\lambda+\mu)\operatorname{rot} V+(\lambda+2\mu)\Delta u+\omega^2 u=-(\lambda+2\mu)\operatorname{grad}\delta(x-x_0), \qquad (23)$$

where $V=\operatorname{rot} u$.

We shall look for a particular solution of this equation in the form of

$$u=\operatorname{grad} f\in\mathcal{D}_2', \quad f=f(x_1, x_2) \qquad (24)$$

which means that

$$V=\operatorname{rot} u=\operatorname{rot}\operatorname{grad} f=0\in\mathcal{D}_3'$$

since

$$(\operatorname{rot}\operatorname{grad} f, \varphi)=(\operatorname{grad} f, \operatorname{rot}\varphi)=-(f, \operatorname{div}\operatorname{rot}\varphi)=$$

$$=-(f, 0)=0 \quad \forall\varphi\in\mathcal{D}_3$$

rules (A) and (C) applied here.

Putting (24) into (23) we conclude that the equation will be satisfied when the distribution f satisfies the equation

$$\Delta f+k^2 f=-\delta(x-x_0), \qquad (25)$$

where $k^2=\dfrac{\omega^2}{\lambda+2\mu}$.

On the basis of the formulas given in paper [1], page 165, we find that the particular solutions of (25) are the functions

$$f_1=\frac{i}{4}H_0^{(1)}(k|x-x_0|), \quad f_2=-\frac{i}{4}H_0^{(2)}(k|x-x_0|),$$

where

$$H_0^{(1)}(z)=J_0(z)+iY_0(z),$$
$$H_0^{(2)}(z)=J_0(z)-iY_0(z),$$

$J_0(z)$ and $Y_0(z)$ denote Bessel functions of the first and second kind, respectively, while $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions.

If we put

$$u=\frac{i}{4}\operatorname{grad} H_0^{(1)}(kr), \quad \text{where } r=|x-x_0|=\sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2}$$

then

$$Au=-(\lambda+2\mu)\operatorname{grad}\delta(x-x_0).$$

Hence, for each smooth function φ with $\operatorname{supp}\varphi\subset\bar{D}$, in case of $x_0\in D$, we obtain

$$\begin{aligned}(Au,\varphi)&=-(\lambda+2\mu)(\operatorname{grad}\delta(x-x_0),\varphi)=(\lambda+2\mu)(\delta(x-x_0),\operatorname{div}\varphi)=\\&=(\lambda+2\mu)\operatorname{div}\varphi(x_0)\end{aligned} \tag{26}$$

and $(Au,\varphi)=0$ when $x_0\notin\bar{D}$.

On the other hand in view of formula (22) for φ we have

$$(Au,\varphi)=(uA\varphi)=-\oint_L\left\{\left[(\lambda+\mu)(n\times\operatorname{rot}\varphi)+(\lambda+2\mu)\frac{\partial\varphi}{\partial n}\right]u-(\lambda+2\mu)\frac{\partial u}{\partial n}\varphi\right\}ds. \tag{27}$$

By comparison of formula (26) with (27) we obtain

$$-(\lambda+2\mu)\operatorname{div}\varphi(x_0)=\oint_L\left[(T\varphi+\lambda N\varphi)u-(\lambda+2\mu)\frac{\partial u}{\partial n}\varphi\right]ds \tag{28}$$

when $x_0\in D$,

$$0=\oint_L\left[(T\varphi+\lambda N\varphi)u-(\lambda+2\mu)\frac{\partial u}{\partial n}\varphi\right]ds \tag{29}$$

provided that $x_0\notin\bar{D}$.

Using the formulas for the derivatives of Hankel functions we have

$$\frac{\partial H_0^{(1)}(kr)}{\partial x_i}=-kH_1^{(1)}(kr)\frac{x_i-x_i^0}{r} \quad (i=1,2)$$

whence formula (28) can be written as

$$\begin{aligned}\operatorname{div}\varphi(x_0)=\frac{ik}{4(\lambda+2\mu)}\oint_L\Bigg[&(T\varphi+\lambda N\varphi)H_1^{(1)}(kr)\frac{x-x_0}{r}-\\&-(\lambda+2\mu)\,\varphi\frac{\partial}{\partial n}\left(H_1^{(1)}(kr)\frac{x-x_0}{r}\right)\Bigg]ds\end{aligned} \tag{30}$$

when $x_0\in D$.

In case of $x_0\notin\bar{D}$ we obtain zero at the left side of (30) in the same way as in (29).

In order to find the formulas for the components of the vector $\varphi(x)$ we proceed as follows:

Assuming that the distribution $\theta = \operatorname{div} u$ is known we represent the value of the operator L for the vector distribution $u=(u_1, u_2)$ in the form

$$Lu=(\lambda+\mu)\operatorname{grad}\theta+\mu B u,$$

where $B=\Delta+k_1^2$, $k_1^2=\omega^2/\mu$.

Now for the test vector we have

$$(Lu, \varphi)=(u, L\varphi)=\int_D [(\lambda+\mu)\operatorname{grad}\operatorname{div}\varphi+\mu\Delta\varphi+\omega^2\varphi]\,u\,dx \tag{31}$$

and respectively

$$\begin{aligned}(Lu, \varphi)&=(\lambda+\mu)(\operatorname{grad}\theta, \varphi)+\mu(Bu, \varphi)=-(\lambda+\mu)(\theta, \operatorname{div}\varphi)+\mu(Bu, \varphi)=\\ &=-(\lambda+\mu)(\{\theta\}-(un)\delta_L, \operatorname{div}\varphi)+\mu\left(\{Bu\}-\frac{\partial u}{\partial n}\delta_L-\frac{\partial}{\partial n}(u\delta_L), \varphi\right)=\\ &=\int_D \mu(\Delta u+k_1^2 u)\,\varphi\,dx+\oint_L\left[\left(\mu\frac{\partial\varphi}{\partial n}+(\lambda+\mu)n\operatorname{div}\varphi\right)u-\mu\frac{\partial u}{\partial n}\varphi\right]ds-\\ &\qquad -(\lambda+\mu)\int_D \theta\operatorname{div}\varphi\,dx.\end{aligned} \tag{32}$$

Now we assume that the test vector satifies the equation $L\varphi=0$ and the vector distributions $u=i_p u_p$ satisfy the equations

$$Bu_p=-\delta(x-x_0), \tag{33}$$

where $i_p (p=1, 2)$ are the versors of the coordinate system.

By these assumptions the right side of the formula (31) is equal to zero, and the right side of (32) has the value

$$\oint_L\left[\left(\mu\frac{\partial\varphi_p}{\partial n}+(\lambda+\mu)(ni_p)\operatorname{div}\varphi\right)u_p-\mu\frac{\partial u_p}{\partial n}\varphi_p\right]ds-$$

$$-(\lambda+\mu)\int_D \theta_p\operatorname{div}\varphi\,dx=\begin{cases}\mu\varphi_p(x_0) & \text{when} \quad x_0\in D,\\ 0 & \text{when} \quad x_0\in D'=R^2\setminus\bar{D}.\end{cases}$$

We know that functions of the form

$$u_p^{(1)}=\frac{i}{4}H_0^{(1)}(k_1|x-x_0|),\quad u_p^{(2)}=-\frac{i}{4}H_0^{(2)}(k_1|x-x_0|)$$

satisfy equation (33) (see the solution of equation (25)). Thus if we put

$$u_p=u_p^{(1)}=\frac{i}{4}H_0^{(1)}(k_1|x-x_0|),$$

the following formula can be obtained

$$\frac{i}{4}\oint_L\left[\left(\frac{\partial\varphi_p}{\partial n}+\frac{\lambda+\mu}{\mu}(ni_p)\operatorname{div}\varphi\right)H_0^{(1)}(k_1|x-x_0|)-\varphi_p\frac{\partial}{\partial n}H_0^{(1)}(k_1|x-x_0|)\right]ds-$$

$$-i\frac{\lambda+\mu}{4\mu}\int_D \operatorname{div}\varphi(x)\frac{\partial}{\partial x_p}H_0^{(1)}(k_1|x-x_0|)dx=\begin{cases}\varphi_p(x_0) & \text{when}\quad x_0\in D\\ 0 & \text{when}\quad x_0\in D'.\end{cases} \tag{34}$$

4. The exact solutions of certain plane problems

Formulas (30) and (34) are also valid for the unbounded domains if the vector function φ satisfies some adequate radiation condition.

Using the above formulas we can easily get the exact solutions of some problems for a half-plane. Now we consider the half-plane $x_2\geqslant 0$ and we find the regular solution of the plane dynamic problem satisfying one of the following boundary conditions;

1) there is given a tangent component of the displacement vector φ and a normal component of the stress vector T

$$\varphi_1|_{x_2=0}=f_1(x_1),\quad (T\varphi)_2|_{x_2=0}=g_2(x_1), \tag{B$_1$}$$

2) there is given a normal component of the displacement vector φ and a tangent component of the stress vector $T\varphi$

$$\varphi_2|_{x_2=0}=f_2(x_1),\quad (T\varphi)_1|_{x_2=0}=g_1(x_1). \tag{B$_2$}$$

In the problems considered, the outside-normal vector $n=-i_2$, therefore $\partial/\partial n=-\partial/\partial x_2$.

Let x_0 belong to the upper half-plane. If we put in formula (30) the point x_0^* symmetric to x_0 with respect to the x_1 axis as an external point, we obtain

$$\operatorname{div}\varphi(x_0)=\frac{ik}{4(\lambda+2\mu)}\int_{-\infty}^{+\infty}\left[(T\varphi+\lambda N\varphi)\,H_1^{(1)}(kr_0)\,\frac{(x_1-x_1^0)i_1-x_2^0 i_2}{r_0}+\right.$$

$$\left.+\varphi\frac{\partial}{\partial x_2}\left(H_1^{(1)}(kr)\frac{(x_1-x_1^0)i_1+(x_2-x_2^0)i_2}{r}\right)\bigg|_{x_2=0}\right]dx_1, \tag{35}$$

$$0=\frac{ik}{4(\lambda+2\mu)}\int_{-\infty}^{+\infty}\left[(T\varphi+\lambda N\varphi)H_1^{(1)}(kr_0)\frac{(x_1-x_1^0)i_1+x_2^0 i_2}{r_0}+\right.$$

$$\left.+\varphi\frac{\partial}{\partial x_2}\left(H_1^{(1)}(kr^*)\frac{(x_1-x_1^0)i_1+(x_2+x_2^0)i_2}{r^*}\right)\bigg|_{x_2=0}\right]dx_1, \tag{36}$$

where

$$r=|x-x_0|=\sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2},\quad r^*=|x-x_0^*|=\sqrt{(x_1-x_1^0)^2+(x_2+x_2^0)^2}$$

$$r_0=r|_{x_2=0}=r^*|_{x_2=0}=\sqrt{(x_1-x_1^0)^2+(x_2^0)^2}.$$

In case of conditions (B_1) being fulfilled we subtract equations (35) and (36) and taking into account the condition

$$\left.\frac{\partial \varphi_1}{\partial x_1}\right|_{x_2=0} = f_1'(x_1)$$

with the assumption of sufficient regularity of the boundary conditions we have

$$\operatorname{div}\varphi = -\frac{ik}{2}\int_{-\infty}^{+\infty}\left[\frac{g_2+\lambda f_1'}{\lambda+2\mu}\frac{x_2^0}{r_0}H_1^{(1)}+f_1\frac{x_2^0}{r_0}(x_1-x_1^0)\frac{d}{dr_0}\left(\frac{1}{r_0}H_1^{(1)}\right)\right]dx_1. \tag{37}$$

In case of conditions (B_2) adding equations (35) and (36) we obtain respectively

$$\operatorname{div}\varphi = \frac{ik}{2}\int_{-\infty}^{+\infty}\left[\frac{g_1-\lambda f_2'}{\lambda+2\mu}\frac{x_1-x_1^0}{r_0}H_1^{(1)}+f_2\frac{(x_2^0)^2}{r_0}\frac{d}{dr_0}\left(\frac{1}{r_0}H_1^{(1)}\right)+f_2\frac{1}{r_0}H_1^{(1)}\right]dx_1. \tag{38}$$

Having found $\operatorname{div}\varphi$ we can start the construction of the displacement vector φ using formulas (34). Similarly to the above procedure we can apply the symmetric-point method to obtain the formulas for displacements.

In case of boundary conditions (B_1) these formulas are of the form

$$\varphi_1(x_0) = \frac{ik_1x_2^0}{2}\int_{-\infty}^{+\infty}f_1H_1^{(1)}\frac{1}{r_0}dx_1+ik_1\frac{\lambda+\mu}{4\mu}\int_0^{\infty}dx_2\int_{-\infty}^{+\infty}\left[\frac{1}{r}H_1^{(1)}(k_1r)-\right.$$
$$\left.-\frac{1}{r^*}H_1^{(1)}(k_1r^*)\right](x_1-x_1^*)\operatorname{div}\varphi\,dx_1,$$

$$\varphi_2(x_0) = -\frac{ik_1}{2}\int_{-\infty}^{+\infty}\left(f_1'-\frac{1}{\mu}g_2\right)H_0^{(1)}dx_1+ik_1\frac{\lambda+\mu}{4\mu}\int_0^{\infty}dx_2\int_{-\infty}^{+\infty}\left[\frac{x_2-x_2^0}{r}H_1^{(1)}(k_1r)+\right.$$
$$\left.+\frac{x_2+x_2^0}{r^*}H_1^{(1)}(k,r^*)\right]\operatorname{div}\varphi\,dx_1.$$

In case of the conditions (B_2) we have respectively

$$\varphi_1(x_0) = \frac{ik_1}{2\mu}\int_{-\infty}^{+\infty}(g_1+\mu f_2')H_0^{(1)}dx_1+ik_1\frac{\lambda+\mu}{4\mu}\int_0^{\infty}dx_2\int_{-\infty}^{+\infty}\left[\frac{1}{r}H_1^{(1)}(k_1r)+\right.$$
$$\left.+\frac{1}{r^*}H_1^{(1)}(k_1r^*)\right](x_1-x_1^0)\operatorname{div}\varphi\,dx_1,$$

$$\varphi_2(x_0) = \frac{ik_1x_2^0}{2}\int_{-\infty}^{+\infty}f_2\frac{1}{r_0}H_1^{(1)}dx_1+ik_1\frac{\lambda+\mu}{4\mu}\int_0^{\infty}dx_2\int_{-\infty}^{+\infty}\left[\frac{x_2-x_2^0}{r}H_1^{(1)}(k_1r)-\right.$$
$$\left.-\frac{x_2+x_2^0}{r^*}H_1^{(1)}(k_1r^*)\right]\operatorname{div}\varphi\,dx_1.$$

Received by the Editor, January 1982.

References

[1] V. S. Vladimirov, *Uravneniya matematicheskoi fiziki*, Moscow 1967.
[2] J. Ryterski, *Równania i przekształcenia całkowe w zastosowaniu do zagadnień teorii sprężystości*. Trans. IF-FM, vol. 59, 1972.
[3] V. D. Kupradze, *Nekotorye zadachi termouprugosti reshaemye v kvadraturakh*. Differentsial'nye uravneniya, vol. V, no. 10, Moscow 1969.
[4] E. I. Obolashvilli, *Effektivnoe reshenie nekotorykh prostranstvennykh zadach teorii uprugosti*. Revue Roumaine de Mathematiques pures et appliques, vol. 11, no. 8, 1966, pp. 965 - 972.

Rozwiązania pewnych zagadnień płaskich drgań ustalonych ośrodka sprężystego

Streszczenie

W pracy rozważa się zagadnienie płaskich ustalonych drgań izotropowych jednorodnego ośrodka sprężystego. Problem rozpatruje się w zakresie teorii dystrybucji. Na wstępie poświęca się uwagę dystrybucjom wektorowym i znajduje odpowiedniki wzorów Greena. Zastosowana metoda rozwiązywania zagadnień teorii sprężystości bazuje na idei przedstawionej w pracy [2]. Pozwoliło to na uzyskanie rozwiązań w postaci zamkniętej pewnych zagadnień szczególnych w obszarach nieograniczonych.

Решения некоторых вопросов плоских устойчивых колебаний упругой среды

Резюме

В работе рассуждается плоская задача устойчивых колебаний изотропной однородной упругой среды. Вопрос рассматривается в переделах теории дистрибуции. Во вступлении сосредотачивается внимание на проблеме векторных дистрибуций и находятся эквиваленты формул Грина. Примененный метод решения задач теории упругости основывается на идеи представленной в труде [2]. Это позволило получить решения в замкнутой форме некоторых особенных задач в неограниченных областях.