PRACE INSTYTUTU MASZYN PRZEPŁYWOWYCH
1983 Zeszyt 86

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On the Potential Boundary Load in the Nonlinear Theory of Thin Shells*

The formula for the work of a stationary force field loading the lateral boundary surface of the shell has been derived. The transformation rules for the resultant boundary force and the resultant boundary static moment as well as the general form of the external virtual work have been given. Finally, the similar transformation rule for the boundary moment of forces and the formula for the work performed by this moment on the rotation of the boundary element have been derived.

1. Introduction

Variational formulation of the theory of thin elastic shells is found to be convenient in the theoretical respect as well as in the numerical one. However, derivation of the appropriate variational functionals requires the knowledge of the strain energy function [1] and of the explicit form of potentials of the external loads applied to the middle surface and the lateral boundary surface of the shell.

Usually determination of such a potential for given external middle-surface load (or the proof for its nonexistence) does not cause any trouble. On the other hand external boundary loads appear in the shell theory in the form of resultant boundary forces and moments acting along the boundary curve. While determination of the potential of the resultant boundary forces consists in integration of the potential of external boundary loads (if it exists) across the shell thickness the same task for the resultant boundary moment acting along the deformed boundary curve is not so trivial. This is so because its definition makes it dependent on the total finite rotation of the boundary element. Therefore, in the case of unrestricted rotations the potential of the resultant boundary moment has to depend on the deformation of the shell and its definition ought to be connected with the potential of the resultant boundary forces, as both these potentials result from the potential of the external continuous loads acting on the lateral boundary surfaces of the shell.

When restricting considerations to at most moderate rotations, the resultant boundary moment induced by constant external boundary load may be treated (within the error of the theory) as independent of the boundary deformation [16]. In such a restricted case it is justified to assume a simple form of the potential of the moment, in analogy to the one

* Praca wykonana w ramach problemu węzłowego 05.12 „Wytrzymałość i optymalizacja konstrukcji maszynowych i budowlanych".

assumed in the linear shell theory. This allows to derive a set of appropriate functionals [2] enabling to formulate different variational principles.

In the shell literature the question of existence of the resultant boundary moment potential has not been discussed for the general case of unrestricted rotations. For instance, Simmonds and Danielson [3, 4] assumed a priori the existence of such potential without giving its explicit form. In papers [5–9] only incremental forms of the work performed by boundary forces and moments in the Eulerian description were given, but without presenting a formula for the total work of these quantities during the process of deformation. In [10] the work of the resultant boundary moment was assumed as a product of the moment and a boundary rotation parameter, but it was not stated what additional conditions had to be imposed on the external continuous boundary loads to ensure correctness of such a simple formula.

In the present paper a general form of the potential has been derived for the resultant boundary forces and moments equivalent to the external continuous boundary loads acting upon the lateral boundary surfaces of the shell. The boundary loads are assumed to be constant during the deformation process. In particular, appropriate transformation rules for the resultant force and the resultant static moment of the boundary loads have been expressed in terms of those resultant quantities. A transformation rule for the resultant boundary moment, such that the assumption about the constancy of the boundary loads during deformation is satisfied, has been determined. The work of the resultant moment has also been derived by direct integration of the appropriate incremental expression. The formula obtained in such an alternative way coincides with the one obtained by integration across the shell thickness of the elementary works of the boundary loads. The paper is supplemented by an appendix dealing with the work performed by a moment of a force field in the rotational motion of a rigid body, where an analogy between the finite rotation of the shell boundary element and the rigid body rotational motion has been discussed.

2. The lateral shell boundary surface and its deformation

The notation used in what follows coincides with that used in [8, 9, 11]. The quantities associated with the deformed shell configuration are provided with overbars.

By the boundary surface $\partial\mathscr{P}$ of the undeformed shell we understand a rectilinear surface which is orthogonal to the middle surface $\mathscr{M}$ of the shell along the boundary curve C. Its arbitrary point $P \in \partial\mathscr{P}$ can be located by a position vector

$$\mathbf{p}(s, z)=\mathbf{r}(s)+z\mathbf{n}(s), \quad -\frac{h}{2} \leqslant z \leqslant \frac{h}{2}, \tag{2.1}$$

where $\mathbf{r}(s)$ is a position vector of the curve C, $\mathbf{n}(s)$ – a unit vector orthogonal to $\mathscr{M}$ along C, z – distance from C in the direction of $\mathbf{n}$, and s denotes the length measure along C.

After deformation compatible with $K-L$ constraints the position vector of the deformed lateral boundary surface takes the form

$$\bar{\mathbf{p}}(\bar{s}, z)=\bar{\mathbf{r}}(\bar{s})+z\bar{\mathbf{n}}(\bar{s}), \tag{2.2}$$

and between the length measures $\bar{s}$ and s along $\bar{C}$ and C, respectively, the following relation holds true

$$d\bar{s}=\bar{a}_t ds, \quad \bar{a}_t=\sqrt{1+2\gamma_{tt}}, \quad \gamma_{tt}=\gamma_{\alpha\beta}t^\alpha t^\beta, \tag{2.3}$$

where $\gamma_{\alpha\beta}$ stands for the components of the strain tensor of the surface $\mathscr{M}$.

With the boundary curve C we may associate an orthonormal triad $\mathbf{v}, \mathbf{t}, \mathbf{n}$, where

$$\mathbf{t}=\frac{d\mathbf{r}}{ds}, \quad \mathbf{v}=\mathbf{t}\times\mathbf{n}. \tag{2.4}$$

After the deformation of the boundary the tangent unit vector $\mathbf{t}$ is transformed to the non-unit, but still tangent to the curve $\bar{C}$, vector $\bar{\mathbf{a}}_t=d\bar{\mathbf{r}}/ds$ with components [9, 11]

$$\bar{\mathbf{a}}_t=c_v\mathbf{v}+c_t\mathbf{t}+c\mathbf{n}. \tag{2.5}$$

Similar orthonormal triad $\bar{\mathbf{v}}, \bar{\mathbf{t}}, \bar{\mathbf{n}}$, associated with the curve $\bar{C}$ is defined by the relations

$$\bar{\mathbf{v}}=\frac{1}{\bar{a}_t}\bar{\mathbf{a}}_v=\frac{1}{\bar{a}_t}\bar{\mathbf{a}}_t\times\mathbf{n}, \quad \bar{\mathbf{t}}=\frac{1}{\bar{a}_t}\bar{\mathbf{a}}_t. \tag{2.6}$$

This may also be written in a much shorter way with the aid of the total rotation tensor R_t of the boundary element

$$\bar{\mathbf{v}}=\mathbf{R}_t\mathbf{v}, \quad \bar{\mathbf{t}}=\mathbf{R}_t\mathbf{t}, \quad \bar{\mathbf{n}}=\mathbf{R}_t\mathbf{n}, \tag{2.7}$$

$$\mathbf{R}_t=\frac{1}{\bar{a}_t}(\bar{\mathbf{a}}_v\otimes\mathbf{v}+\bar{\mathbf{a}}_t\otimes\mathbf{t})+\bar{\mathbf{n}}\otimes\mathbf{n}. \tag{2.8}$$

Such deformation of the lateral boundary surface implies the existence of a displacement field $\mathbf{v}$ determined by

$$\mathbf{v}(s,\mathfrak{z})=\bar{\mathbf{p}}-\mathbf{p}=\mathbf{u}+\mathfrak{z}(\bar{\mathbf{n}}-\mathbf{n})=\mathbf{u}+\mathfrak{z}(\mathbf{R}_t-\mathbf{1})\mathbf{n}, \tag{2.9}$$

where

$$\mathbf{u}=\bar{\mathbf{r}}-\mathbf{r}=u_v\mathbf{v}+u_t\mathbf{t}+w\mathbf{n} \tag{2.10}$$

denotes the displacement field of the middle surface of the shell along the boundary curve C and $\mathbf{1}$ stands for the metric tensor of the three-dimensional Euclidean space defined by

$$\mathbf{1}=\mathbf{v}\otimes\mathbf{v}+\mathbf{t}\otimes\mathbf{t}+\mathbf{n}\otimes\mathbf{n}=\bar{\mathbf{v}}\otimes\bar{\mathbf{v}}+\bar{\mathbf{t}}\otimes\bar{\mathbf{t}}+\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}. \tag{2.11}$$

On the surface $\partial\mathscr{P}$ (or $\partial\bar{\mathscr{P}}$) we may now specify a family of straight lines, determined by the equations $s=\text{const}$ (or $\bar{s}=\text{const}$) and parametrized by $\mathfrak{z}$, and a family of curves C^* (or $\bar{C}^*$) equidistant from the curve C (or $\bar{C}$), determined by the equations $\mathfrak{z}=\text{const}$, whose parameters are denoted by s^* (or $\bar{s}^*$). Generally, the parameters s^* (or $\bar{s}^*$) are not measures of length for the curves C^* (or $\bar{C}^*$). These two families may be treated as curvilinear coordinates on $\partial\mathscr{P}$ (or $\partial\bar{\mathscr{P}}$).

The area element of the surface $\partial\mathscr{P}$, bounded by the curves $s^*=\text{const}$, $s^*+ds^*=\text{const}$, $\mathfrak{z}=\text{const}$ and $\mathfrak{z}+d\mathfrak{z}=\text{const}$, is determined by

$$d\mathbf{A}^*=dA^*\mathbf{v}^*=d\mathbf{s}^*\times\mathbf{n}d\mathfrak{z}=\left(\mathbf{v}+\mathfrak{z}\frac{d\mathbf{n}}{ds}\times\mathbf{n}\right)ds\,d\mathfrak{z}, \tag{2.12}$$

where we have employed the following relation

$$ds^* = ds^* \mathbf{t}^* = \left(\mathbf{t} + z\frac{d\mathbf{n}}{ds}\right) ds. \tag{2.13}$$

In the same way we will find the area element of the deformed surface $\partial\bar{\mathscr{P}}$ to be

$$d\bar{\mathbf{A}}^* = \left(\bar{\mathbf{v}} + z\frac{d\bar{\mathbf{n}}}{d\bar{s}} \times \bar{\mathbf{n}}\right) d\bar{s}\, dz = \left(\bar{\mathbf{a}}_v + z\frac{d\bar{\mathbf{n}}}{ds} \times \bar{\mathbf{n}}\right) ds\, dz. \tag{2.14}$$

We will take advantage of these relations in the following parts of this paper.

3. External boundary load

Let $\bar{\mathbf{f}} = \bar{\mathbf{f}}(\bar{s}, z, \mathbf{v})$ denote the external boundary load acting on the surface $\partial\bar{\mathscr{P}}$ and measured per its unit area.

Let us introduce a function

$$g(s, z) = \frac{d\bar{A}^*}{dA^*} = \frac{\left|\bar{\mathbf{a}}_v + z\dfrac{d\bar{\mathbf{n}}}{ds} \times \bar{\mathbf{n}}\right|}{\left|\mathbf{v} + z\dfrac{d\mathbf{n}}{ds} \times \mathbf{n}\right|} \tag{3.1}$$

describing the change of area during the deformation of the lateral shell boundary surface. Then the force acting on the area element $d\bar{A}^*$ can be determined by the formula

$$d\mathbf{S}^* = \bar{\mathbf{f}}\, d\bar{A}^* = \bar{\mathbf{f}} g\, dA^*. \tag{3.2}$$

Henceforth, we will make the following assumption about the field $\bar{\mathbf{f}}$:

during deformation of the boundary surface from the undeformed configuration $\partial\mathscr{P}$ to the deformed configuration $\partial\bar{\mathscr{P}}$, described by the displacement field $\mathbf{v}$, the field $\bar{\mathbf{f}}$ of the external boundary loads varies in such a manner that the force $d\mathbf{S}^*$ acting on the area element $d\bar{A}^*$ remains always constant. This may also be formulated in the following way:

$$d\mathbf{S}^* = \bar{\mathbf{f}} g\, dA^* = \mathbf{f}\, dA^* = \text{const}, \tag{3.3}$$

where

$$\mathbf{f} = \bar{\mathbf{f}} g = \bar{\mathbf{f}}|_{\mathbf{v}=0} = \text{const} \tag{3.4}$$

denotes the value of the external boundary load at the beginning of the deformation process.

Employing the notion of gradient we also have [13]

$$\frac{d(\bar{\mathbf{f}} g)}{d\mathbf{v}} = \mathbf{0}. \tag{3.5}$$

The field $\bar{\mathbf{f}}$ can be characterized by certain quantities, resultant with respect to the boundary curve $\bar{C}$, namely the resultant boundary force and the resultant boundary static moments of arbitrary orders. Within the K – L type theory of shells only the resultant boundary force and the resultant boundary static moment of the first order are taken into consideration, i.e.

$$\bar{\mathbf{F}} d\bar{s} = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \bar{\mathbf{f}}\, d\bar{A}^*, \quad \bar{\mathbf{H}}\, d\bar{s} = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \bar{\mathbf{f}} z\, d\bar{A}^*. \tag{3.6}$$

The assumption (3.3) or (3.5) implies the following transformation rules for $\bar{\mathbf{F}}$ and $\bar{\mathbf{H}}$ during the deformation

$$\begin{aligned} \bar{\mathbf{F}} d\bar{s} &= \bar{\mathbf{F}} \bar{a}_t\, ds = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \bar{\mathbf{f}}\, d\bar{A}^* = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \mathbf{f}\, dA^* = \mathbf{F}\, ds, \\ \bar{\mathbf{H}} d\bar{s} &= \bar{\mathbf{H}} \bar{a}_t\, ds = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \bar{\mathbf{f}} z\, d\bar{A}^* = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \mathbf{f} z\, dA^* = \mathbf{H} ds \end{aligned} \tag{3.7}$$

or

$$\bar{\mathbf{F}} \bar{a}_t = \mathbf{F}, \quad \bar{\mathbf{H}} \bar{a}_t = \mathbf{H}, \tag{3.8}$$

where $\mathbf{F}$ and $\mathbf{H}$ are the corresponding resultant quantities at the beginning of the deformation process, defined with respect to the curve C.

4. The work of external boundary loads and its variation

The work L of the external boundary loads, performed during the deformation process may be determined as an integral

$$L = \oint_C W\, ds = \oint_{\bar{C}} \bar{W}\, d\bar{s}, \tag{4.1}$$

where W and $\bar{W}$ denote the work per unit length of C and $\bar{C}$, respectively. The relations (4.1) and (2.3) imply that

$$W = \bar{W} \bar{a}_t. \tag{4.2}$$

On the other hand, by virtue of the definition of work, the following holds true:

$$L = \iint_{\partial\bar{\mathscr{P}}} \int_0^{\mathbf{v}} \bar{\mathbf{f}}\, d\bar{A}^* \cdot d\mathbf{v} = \iint_{\partial\mathscr{P}} \int_0^{\mathbf{v}} \mathbf{f}\, dA^* \cdot d\mathbf{v}. \tag{4.3}$$

Taking into account the assumption (3.5) we may perform partial integration in (4.3) with respect to displacements, which leads to the formula

$$L = \iint_{\partial\bar{\mathscr{P}}} \mathbf{v} \cdot \bar{\mathbf{f}}\, d\bar{A}^* = \iint_{\partial\mathscr{P}} \mathbf{v} \cdot \mathbf{f}\, dA^*. \tag{4.4}$$

Then, after substitution of Eq. (2.9) into (4.4) we obtain

$$L = \oint_{\bar{C}} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} [\mathbf{u} + z(\bar{\mathbf{n}} - \mathbf{n})] \cdot \bar{\mathbf{f}}\, d\bar{A}^* = \oint_{\bar{C}} [\mathbf{u} \cdot \bar{\mathbf{F}} + (\bar{\mathbf{n}} - \mathbf{n}) \cdot \bar{\mathbf{H}}]\, d\bar{s} = \oint_C [\mathbf{u} \cdot \mathbf{F} + (\bar{\mathbf{n}} - \mathbf{n}) \cdot \mathbf{H}]\, ds. \tag{4.5}$$

This implies the following expressions for $\bar{W}$ and W

$$\bar{W} = \mathbf{u} \cdot \bar{\mathbf{F}} + (\bar{\mathbf{n}} - \mathbf{n}) \cdot \bar{\mathbf{H}} = \mathbf{u} \cdot \bar{\mathbf{F}} + (\mathbf{R}_t - \mathbf{1}) \cdot (\bar{\mathbf{H}} \otimes \mathbf{n}), \tag{4.6}$$

$$W = \mathbf{u} \cdot \mathbf{F} + (\bar{\mathbf{n}} - \mathbf{n}) \cdot \mathbf{H} = \mathbf{u} \cdot \mathbf{F} + (\mathbf{R}_t - \mathbf{1}) \cdot (\mathbf{H} \otimes \mathbf{n}), \tag{4.7}$$

where the dots in the last terms denote scalar products in the tensor space. For the second-order tensors **A** and **B** the scalar product is obtained according to the rule $\mathbf{A}\cdot\mathbf{B}=\mathrm{tr}\,[\mathbf{A}^T\mathbf{B}]=A^{ij}B_{ij}$. The scalar product operation should be performed after completing the tensor multiplication, i.e. $\mathbf{AB}\cdot\mathbf{C}\equiv(\mathbf{AB})\cdot\mathbf{C}$.

The scalar form of (4.7) is as follows

$$W=u_\nu F_\nu+u_t F_t+wF+n_\nu H_\nu+n_t H_t+(n-1)H. \tag{4.8}$$

The function $-W$ may be interpreted as a potential of the generalized boundary forces **F** and **H** in a six-dimensional space of states $(\mathbf{u}, \bar{\mathbf{n}}-\mathbf{n})$ describing the deformation of the surface $\partial\mathscr{P}$. In this sense the generalized forces **F** and **H** may be called the potential forces, since the following is satisfied

$$\frac{\partial W}{\partial u_\nu}=F_\nu,\quad \frac{\partial W}{\partial u_t}=F_t,\quad \frac{\partial W}{\partial w}=F,\quad \frac{\partial W}{\partial n_\nu}=H_\nu,\quad \frac{\partial W}{\partial n_t}=H_t,\quad \frac{\partial W}{\partial n}=H. \tag{4.9}$$

In the K–L type shell theory four independent parameters only describe completely the deformation of the surface $\partial\mathscr{P}$. As was shown in [11], for the Lagrangian shell theory, u_ν, u_t, w and n_ν are these parameters, while the remaining two can be expressed in terms of them by the relations

$$\begin{aligned} n_t&=-\frac{1}{c_t^2+c^2}\left(c_t c_\nu n_\nu+c\sqrt{\bar{a}_t(1-n_\nu^2)-c_\nu^2}\right),\\ n&=-\frac{1}{c_t^2+c^2}\left(cc_\nu n_\nu-c_t\sqrt{\bar{a}_t(1-n_\nu^2)-c_\nu^2}\right). \end{aligned} \tag{4.10}$$

After substitution of (4.10) into (4.8) W becomes dependent on four parameters only The variation δW may be obtained directly from (4.8), after substitution of (4.10) in there, but it will be much easier to make use of the relations derived in [11]. We obtain

$$\delta W=F_\nu\delta u_\nu+F_t\delta u_t+F\delta w+H_\nu\delta n_\nu+H_t\delta n_t+H\delta n, \tag{4.11}$$

where

$$\begin{aligned} \delta n_t&=d_\nu\delta u_\nu+d_t\delta u_t+d\delta w+f\delta n_\nu+g_\nu\frac{d}{ds}\delta u_\nu+g_t\frac{d}{ds}\delta u_t+g\frac{d}{ds}\delta w,\\ \delta n&=h_\nu\delta u_\nu+h_t\delta u_t+h\delta w+k\delta n_\nu+r_\nu\frac{d}{ds}\delta u_\nu+r_t\frac{d}{ds}\delta u_t+r\frac{d}{ds}\delta w. \end{aligned} \tag{4.12}$$

All the coefficients appearing in equations (4.12) were explained in [11]. After substitution of (4.12) into (4.11) and some transformations we get the following expression for δW

$$\begin{aligned} \delta W=&\left[F_\nu+H_t d_\nu+Hh_\nu-\frac{d}{ds}(H_t g_\nu+Hr_\nu)\right]\delta u_\nu+\left[F_t+H_t d_t+Hh_t-\frac{d}{ds}(H_t g_t+Hr_t)\right]\delta u_t+\\ &+\left[F+H_t d+Hh-\frac{d}{ds}(H_t g+Hr)\right]\delta w+[H_\nu+H_t f+Hk]\delta n_\nu+\\ &+\frac{d}{ds}[(H_t g_\nu+Hr_\nu)\delta u_\nu+(H_t g_t+Hr_t)\delta u_t+(H_t g+Hr)\delta w], \end{aligned} \tag{4.13}$$

which may be expressed in a more compact form

$$\delta W = \mathbf{P}^* \cdot \delta \mathbf{u} + M^* \delta n_\nu + \frac{d}{ds} (\mathbf{F}^* \cdot \delta \mathbf{u}).$$

Making use of the above relations we can easily find the expression for the external virtual work

$$EVW = \iint\limits_{\mathcal{M}} \mathbf{p} \cdot \delta \mathbf{u} \, dA + \oint\limits_{C} \left[\mathbf{P}^* \cdot \delta \mathbf{u} + M^* \delta n_\nu + \frac{d}{ds} (\mathbf{F}^* \cdot \delta \mathbf{u}) \right] ds =$$

$$= \iint\limits_{\mathcal{M}} \mathbf{p} \cdot \delta \mathbf{u} \, dA + \int\limits_{C} [\mathbf{P}^* \cdot \delta \mathbf{u} + M^* \delta n_\nu] \, ds + \sum_{i}^{N} \mathbf{F}_i^*(S_i) \cdot \delta \mathbf{u}_i(S_i), \tag{4.14}$$

where now $\mathbf{p}$ denotes a conservative middle-surface dead-type load and the last term describes the variation of the work performed in points of discontinuity of the boundary forces or in the corners of the boundary curve C. The structure of the expression for EVW is analogous to that of the internal virtual work [11].

5. The resultant boundary moment

In the theory of thin shells the continuous boundary load, acting on the lateral boundary surface, is usually described by the resultant boundary force $\overline{\mathbf{F}}$ and the resultant boundary moment $\overline{\mathbf{k}}$, both measured per unit length of the deformed boundary. $\overline{\mathbf{k}}$ is defined as the following vector product

$$\overline{\mathbf{k}} d\bar{s} = \int\limits_{-\frac{1}{2}h}^{\frac{1}{2}h} \zeta \overline{\mathbf{n}} \times \overline{\mathbf{f}} \, d\bar{A}^* \equiv \overline{\mathbf{n}} \times \overline{\mathbf{H}} d\bar{s}. \tag{5.1}$$

This yields the relation

$$\overline{\mathbf{k}} \bar{a}_t = \overline{\mathbf{n}} \times \mathbf{H}. \tag{5.2}$$

After completing the vector multiplication in (5.1) we also get

$$\overline{\mathbf{k}} = -\bar{H}_t \overline{\mathbf{v}} + \bar{H}_\nu \overline{\mathbf{t}}. \tag{5.3}$$

It can be observed from the above relations that during deformation the vector $\overline{\mathbf{k}}$ generally varies in magnitude as well as in direction.

Let us find its transformation rule. We are searching for such a tensor $\mathbf{X}$ that

$$\overline{\mathbf{k}} = \mathbf{X} \mathbf{k}, \tag{5.4}$$

where $\mathbf{k}$ denotes the appropriate resultant boundary moment in the underformed state of the shell. Then we may write

$$\frac{1}{\bar{a}_t} (\overline{\mathbf{n}} \times \mathbf{1}) \mathbf{H} = \mathbf{X} (\mathbf{n} \times \mathbf{1}) \mathbf{H}.$$

The cross product, applied to tensors, denotes vector multiplication performed on the base vectors nearest to it, e.g. for two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ we have

$$\mathbf{A} \times \mathbf{B} = (A^{ij} \mathbf{g}_i \otimes \mathbf{g}_j) \times (B^{km} \mathbf{g}_k \otimes \mathbf{g}_m) = A^{ij} \varepsilon_{jkl} B^{km} \mathbf{g}_i \otimes \mathbf{g}^l \otimes \mathbf{g}_m.$$

Since the vector $\mathbf{H}$ is arbitrary we obtain

$$\bar{\mathbf{k}}=\frac{1}{\bar{a}_t}(\bar{\mathbf{n}}\times\mathbf{1})(\mathbf{n}\times\mathbf{1})^{-1}\mathbf{k} \tag{5.5}$$

or

$$(\bar{\mathbf{n}}\times\mathbf{1})^{-1}\bar{\mathbf{k}}\bar{a}_t=(\mathbf{n}\times\mathbf{1})^{-1}\mathbf{k}=\text{const}. \tag{5.6}$$

These relations establish the transformation rule for the vector of boundary moment $\bar{\mathbf{k}}$, corresponding to the continuous boundary loads satisfying assumption (3.5), during the deformation process.

The vector $\bar{\mathbf{k}}$ may be regarded as an axial vector of a skew-symmetric moment tensor, denoted by $\bar{\mathbf{K}}$ (see [13, 14]). Then the relations between the vector and the tensor are as follows

$$\begin{gathered}\bar{\mathbf{K}}=\bar{\mathbf{k}}\times\mathbf{1},\\ \bar{\mathbf{k}}=\tfrac{1}{2}(\mathbf{1}\times\mathbf{1})\cdot\bar{\mathbf{K}}.\end{gathered} \tag{5.7}$$

At the same time (2.8) and (3.8) imply the following transformation rule for the tensor $\bar{\mathbf{K}}$:

$$\bar{\mathbf{K}}=2(\bar{\mathbf{H}}\otimes\bar{\mathbf{n}})^A=\frac{1}{\bar{a}_t}[(\mathbf{H}\otimes\mathbf{n})\mathbf{R}_t^T-\mathbf{R}_t(\mathbf{n}\otimes\mathbf{H})], \tag{5.8}$$

where $(\mathbf{T})^A$ denotes the skew-symmetric part of a tensor $\mathbf{T}$. Apparently, so defined tensor is a tensor of (A.8) type as derived in the Appendix.

6. Work of the resultant boundary moment

In what follows we shall make use of the following tensor identity: for three arbitrary second-order tensors the formula

$$\mathbf{AB}\cdot\mathbf{C}^T=\mathbf{A}\cdot(\mathbf{BC})^T=\mathbf{1}\cdot(\mathbf{ABC}) \tag{6.1}$$

is true.

Taking into account (A.6) and (5.7) we obtain the formula for the increment $d\bar{W}_R$ of the part of work resulting from rotation

$$d\bar{W}_R=(\bar{\mathbf{n}}\otimes\bar{\mathbf{H}})^A\cdot[(d\mathbf{R}_t^T)\mathbf{R}_t]. \tag{6.2}$$

From (A.4) we know that $(d\mathbf{R}_t^T)\mathbf{R}$ is a skew-symmetric tensor (and the scalar product of antisymmetric and symmetric tensors is zero). This, together with (3.8) and (2.7), allows us to write

$$d\bar{W}_R=\frac{1}{\bar{a}_t}\mathbf{R}_t(\mathbf{n}\otimes\mathbf{H})\cdot[(d\mathbf{R}_t^T)\mathbf{R}_t]. \tag{6.3}$$

Now, according to (4.2) and (6.1), we can make the following transformations:

$$dW_R=(d\mathbf{R}_t^T)\mathbf{R}_t\cdot\mathbf{R}_t(\mathbf{n}\otimes\mathbf{H})=(d\mathbf{R}_t^T)\cdot\mathbf{R}_t^T\mathbf{R}_t(\mathbf{n}\otimes\mathbf{H})=(\mathbf{n}\otimes\mathbf{H})\cdot d\mathbf{R}_t^T. \tag{6.4}$$

As the tensor $(\mathbf{n}\otimes\mathbf{H})$ is constant during deformation we may integrate (6.4) to obtain

$$W_R = \int_{1}^{\mathbf{R}_t^T} (\mathbf{n}\otimes\mathbf{H})\cdot d\mathbf{R}_t^T = (\mathbf{R}_t^T - \mathbf{1})\cdot(\mathbf{n}\otimes\mathbf{H}) = (\bar{\mathbf{n}} - \mathbf{n})\cdot\mathbf{H}\,. \tag{6.5}$$

The result obtained coincides with that obtained in (4.5) from the integration across the shell thickness.

The above considerations lead to the conclusion that within the K−L shell theory the resultant boundary force $\bar{\mathbf{F}}$ and the resultant boundary moment $\bar{\mathbf{k}}$, which obey during deformation the transformation rules (3.7) and (5.5), respectively, are formally equivalent to the continuous boundary surface loads $\bar{\mathbf{f}}$ satisfying the assumption (3.5). This assumption may also be treated as a transformation rule for $\bar{\mathbf{f}}$.

Appendix

Let us discuss a rigid body in a rotational motion around a fixed point 0. Let us also associate with it, at the beginning of the motion, an orthonormal triad $\mathbf{v}_0$, $\mathbf{t}_0$, $\mathbf{n}_0$ attached to the point 0. Then the position of the body after arbitrary time t may be established with the aid of the triad and the rotation tensor $\mathbf{R}(t)$

$$\mathbf{v} = \mathbf{R}\mathbf{v}_0\,,\qquad \mathbf{t} = \mathbf{R}\mathbf{t}_0\,,\qquad \mathbf{n} = \mathbf{R}\mathbf{n}_0\,. \tag{A.1}$$

Let us consider an arbitrary position vector $\mathbf{r}_0$ of the body particles, having its origin at the point 0. During the motion its end-point will move along a certain curve $\mathbf{r}(t)$. The velocity of this point is given by the formula

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d(\mathbf{R}\mathbf{r}_0)}{dt} = \dot{\mathbf{R}}\mathbf{r}_0\,. \tag{A.2}$$

On the other hand we have

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = \boldsymbol{\omega}\times(\mathbf{R}\mathbf{r}_0)\,, \tag{A.3}$$

where $\boldsymbol{\omega}$ is an instantaneous angular velocity vector of the body. In the face of arbitrariness of $\mathbf{r}_0$ we may write [4]

$$\boldsymbol{\omega}\times\mathbf{R} = \dot{\mathbf{R}}$$

or

$$\boldsymbol{\omega}\times\mathbf{1} = \dot{\mathbf{R}}\mathbf{R}^T\,,\qquad \boldsymbol{\omega} = -\tfrac{1}{2}(\mathbf{1}\times\mathbf{1})\cdot(\dot{\mathbf{R}}\mathbf{R}^T)^T\,, \tag{A.4}$$

where $\boldsymbol{\omega}\times\mathbf{1}$ is a skew-symmetric tensor called the spin tensor of a rigid body.

Let the body be subjected to a moment vector $\mathbf{k}(t)$ varying in time and attached to the point 0. Then an increment of work performed by this moment can be defined by the formula

$$dL = \mathbf{k}(t)\cdot\boldsymbol{\omega}(t)\,dt\,. \tag{A.5}$$

After substitution of (A.4) to (A.5) we get

$$dL = -\tfrac{1}{2}\mathbf{k}\,[(\mathbf{1}\times\mathbf{1})\cdot(\dot{\mathbf{R}}\mathbf{R}^T)^T]\,dt = -\tfrac{1}{2}(\mathbf{k})\cdot(d\mathbf{R}\mathbf{R}^T)^T\,, \tag{A.6}$$

where $\mathbf{k}\times\mathbf{1} = \mathbf{K}$ denotes a skew-symmetric moment tensor.

Now let us assume that the moment tensor represents a skew-symmetric part of a tensor $\mathbf{T}$ which changes during the motion according to the following rule

$$\mathbf{T} = \mathbf{R}\mathbf{T}_0\,, \tag{A.7}$$

where $\mathbf{T}_0$ denotes a certain constant tensor, i.e. $d\mathbf{T}_0/dt = \mathbf{0}$. Then we can write:

$$\mathbf{K} = \mathbf{T}^A = \tfrac{1}{2}\,[\mathbf{R}\mathbf{T}_0 - \mathbf{T}_0^T\mathbf{R}^T]\,. \tag{A.8}$$

After substitution of (A.8) to (A.6) we obtain the following relation for the elementary work:

$$dL = -\tfrac{1}{2}\mathbf{T}^A \cdot (d\mathbf{R}\mathbf{R}^T)^T = -\tfrac{1}{2}\mathbf{T} \cdot (d\mathbf{R}\mathbf{R}^T)^T, \tag{A.9}$$

which, after making use of formula (6.1), may be transformed to the form

$$dL = -\tfrac{1}{2}\mathbf{R}^T\mathbf{T} \cdot d\mathbf{R}^T = -\tfrac{1}{2}\mathbf{R}^T\mathbf{R}\mathbf{T}_0 \cdot d\mathbf{R}^T = -\tfrac{1}{2}\mathbf{T}_0 \cdot d\mathbf{R}^T.$$

After integrating

$$L = -\tfrac{1}{2}\int_{\mathbf{R}(t_0)}^{\mathbf{R}(t_1)} \mathbf{T}_0\, d\mathbf{R}^T = -\tfrac{1}{2}\mathbf{T}_0 \cdot (\mathbf{R}_1 - \mathbf{R}_0) \tag{A.10}$$

we obtain the formula for the work in the rotational motion performed by a moment of the form of (A.8).

Thus all moment tensors of the form of (A.8) may be regarded as potential tensors. In particular, in the nonlinear theory of shells, the tensor of the resultant boundary moment, obtained from the external boundary loads satisfying (3.4), belongs to this class.

As an example of application of these relations let us consider a rigid body with one point fixed (let it be the point 0), subjected to a force field **f**, which depend only on the point of the body and do not depend on time. Let us assume next that we know the history of this motion, i.e. the function $\mathbf{R} = \mathbf{R}(t)$ such that $\mathbf{R}(t_0) = \mathbf{1}$.

Due to the lack of translatory motion the only quantity performing work is the moment of forces **f**, determined with respect to the point 0.

The moment tensor is defined by the formula [5]

$$\mathbf{K} = \iiint_{\Omega} [\mathbf{f} \otimes \mathbf{r} - \mathbf{r} \otimes \mathbf{f}]\, dV,$$

where Ω denotes the domain of the body, dV — the element of its volume and $\mathbf{r} = \mathbf{R}\mathbf{r}_0 = \mathbf{r}_0\mathbf{R}^T$ its position vector with the origin at the point 0. Then we may write

$$\mathbf{K} = -2\,[\iiint_{\Omega} \mathbf{r} \otimes \mathbf{f}\, dV\,]^A$$

and thus the moment tensor **K** belongs to the class (A.8). In this case

$$\mathbf{T}_0 = -2\iiint_{\Omega} \mathbf{r}_0 \otimes \mathbf{f}\, dV,$$

and the work is given by the formula

$$L = (\mathbf{R} - \mathbf{1}) \cdot \iiint_{\Omega} \mathbf{f} \otimes \mathbf{r}_0\, dV.$$

The tensor $-\tfrac{1}{2}\mathbf{T}_0$ may be interpreted here as the static moment of the force field **f** with respect to the point of fixation, determined in the initial instant of the motion.

Received by the Editor, January 1981.

References

[1] W. T. Koiter, *A Consistent First Approximation in the General Theory of Thin Elastic Shells*. Proc. IUTAM Symp., „Theory of Thin Shells, Delft 1959.

[2] W. Pietraszkiewicz, R. Schmidt, *Variational principles in the Geometrically Non-linear Theory of Shells Undergoing Moderate Rotations*". Bochum, July 1979 (in print in Ingeneur-Archiv).

[3] J. G. Simmonds, D. A. Danielson, *Nonlinear Shell Theory with a Finite Rotation Vector*. Proc. Kon. Ned. Ak. Wet. B73 (1970), 460 - 478.

[4] J. G. Simmonds, D. A. Danielson, *Nonlinear Shell Theory with Finite Rotation and Stress-Function Vectors*. J. Appl. Mech. 39 (1972). No 4, 1085 - 1090.

[5] W. T. Koiter, *On the Nonlinear Theory of Thin Elastic Shells*. Proc. Kon. Ned. Ak. Wet., Ser. B, vol. 69, 1966, No 1, 1 - 54.

[6] B. Budiansky, *Notes on Nonlinear Shell Theory*. J. Appl. Mech., Trans. ASME, Ser. E. vol. 36, June 1968, 393 - 401.
[7] J. L. Sanders, *Nonlinear Theories for Thin Shells*. Quart. of Appl. Math., Vol. XXI (1963), 21 - 36.
[8] W. Pietraszkiewicz, *Lagrangean Nonlinear Theory of Shells*. Arch. Mech. Stosow., 26 (1974), No 2, 221 - 228.
[9] W. Pietraszkiewicz, *Introduction to the Nonlinear Theory of Shells*. Ruhr-Universität Bochum, Mitteilungen aus dem Institut für Mechanik No 10, Mai 1977, 1 - 154.
[10] K. Z. Galimow, *Osnowy nieliniejnoj tieorii tonkich obołoczek*. Izdatielstwo Kazanskowo Uniwiersitieta, 1975.
[11] W. Pietraszkiewicz, M. Szwabowicz, *Lagrangian Nonlinear Theory of Thin Shells*. Archives of Mechanics (in print).
[12] W. Pietraszkiewicz, *Finite Rotations in Shells*. [In:] *Theory of Shells*. Ed. by W. T. Koiter, G. K. Mikhailov, Proc. 3rd IUTAM Symp., Tbilisi 1978; North-Holland P. Co. Amsterdam 1980, 445 - 471.
[13] D. C. Leigh, *Nonlinear Continuum Mechanics*. Mc Graw-Hill Book Co., N. Y., 1968.
[14] L. Brillouin, *Tensors in Mechanics and Elasticity*. Academic Press, N. Y., London, 1964.
[15] J. Więckowski, *Elementy mechaniki ciała sztywnego w zapisie absolutnym i pewne jej uogólnienia*. Biuletyn IMP PAN 72/736/1973, Gdańsk.
[16] W. Pietraszkiewicz, *Finite Rotations in the Non-linear Theory of Thin Shells*. Wykłady w CISM w Udine ,,Thin Shells'', October 1977 (w druku w Springer-Verlag, Wien).

O potencjalnym obciążeniu brzegowym w nieliniowej teorii cienkich powłok

Streszczenie

Artykuł dotyczy problemu określenia potencjału zewnętrznych obciążeń brzegowych w nieliniowej teorii powłok doznających skończonych obrotów.

W pierwszej części artykułu wyprowadzono wzór na pracę pola sił obciążającego boczną powierzchnię brzegową powłoki. Zakłada się stałość pola sił podczas trwania procesu deformacji. Wielkościami występującymi we wzorze są wypadkowa siła brzegowa i wypadkowy moment statyczny na brzegu, odniesione do jednostki długości nieodkształconej krzywej brzegowej. Podano również reguły transformacji, którym podlegają zmiany tych wielkości podczas odkształcenia. Otrzymany wynik posłużył do wyprowadzenia ogólnej postaci zewnętrznej pracy wirtualnej.

W drugiej części artykułu wyprowadzono podobną regułę transformacji dla wypadkowego brzegowego momentu sił dla pola sił brzegowych spełniającego założenie podane wyżej. Podano wzór na pracę wykonywaną przez ten moment przy obrocie elementu brzegowego. Otrzymany wynik jest zbieżny z wynikiem podanym w pierwszej części pracy.

Artykuł uzupełniono Dodatkiem dotyczącym pracy momentu sił wykonywanej przy ruchu obrotowym ciała sztywnego.

О потенциальной краевой нагрузке в нелинейной теории тонких оболочек

Резюме

Статья касается проблемы нахождения потенциала внешней краевой нагрузки в нелинейной теории оболочек подвергающихся конечным оборотам.

В первой части статьи выведена формула на работу поля сил нагружающего боковую краевую поверхность оболочки. Предполагается постоянство поля сил во время процесса деформации. Величинами выступающими в формуле являются: результирующая краевая сила и результирующий

статический момент на крае, отнесенные к единице длины недеформированной краевой кривой. Даны также принципы трансформации, по которым происходят изменения этих величин во время деформации. Полученный результат послужил для выведения общей формы внешней виртуальной работы.

Во второй части статьи выведен похожий принцип трансформации для результирующего краевого момента сил для поля краевых сил соблюдающего вышеопределенное предположение. Представлена формула на работу производимую этим моментом при обороте краевого элемента. Полученный результат сходим с результатом представленным в первой части статьи.

Статья пополняется приложением касающимся работы момента сил, производимой при вращательном движении твердого тела.