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exist for the publication of theoretical and experimental investigations of all aspects of the mechanics and thermodynamics of fluid-flow with special reference to fluid-flow machinery

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ul. Gen. Józefa Fiszera 14, 80-952 Gdańsk, skr. pocztowa 621, tel. 41-12-71

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JAROSŁAW MIKIELEWICZ, EUGENIUSZ IHNATOWICZ, STANISŁAW GUMKOWSKI

Gdańsk

An Analysis of the Pressure Gradient Effect on the Film Flow and its Stability*

An analysis of the pressure gradient effect on the thin liquid film flowing over the channel wall has been carried out. Wavy film flow and the pressure gradient effect on the film breakdown have also been considered.

Nomenclature

a – wave amplitude, thermal diffusivity,
A – cross-sectional area of the channel,
c – wave velocity,
c_p – isobaric specific heat,
d – channel diameter,
E – total mechanical energy,
e – mechanical energy per unit width,
G – mass flow rate,
I – integral, function of the parameter θ_0
k – wave number,
K – parameter defined by eq. (22),
p – pressure,
R – rivulet radius,
R_g – gas constant,
s – distance between rivulets,
q – mass flow rate per unit width,
t – time,
T – temperature,
u – velocity,
U – channel perimeter,
x, y, z – coordinates,
X – parameter defined by eq. (44),
α – heat transfer coefficient,
δ – water film thickness,
θ_0 – contact angle,
λ – wave length, thermal conductivity,
μ – dynamic viscosity,
ν – kinematic viscosity,
ρ – density,
σ – surface tension,
τ – shear stress,
φ – function defined by eq. (30),
ω – wave frequency,
Nu – Nusselt number,
Re – Reynolds number,
Pr – Prandtl number.

Subscripts

e – equivalent diameter,
f – film,
g – gas,
i – liquid-gas interface,
n – n-th approximation,
0 – equivalent value,
p – pressure,
r – rivulet,
s – surface,
$-$ – mean value,
$+$ – nondimensional parameter.

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1. Introduction

Research on flow, stability and heat transfer of shear driven thin liquid films has been carried out at the Institute of Fluid-Flow Machines for several years [1÷4].

The purpose is to learn the mechanics of the water film flow in the last steam turbine stages. Knowledge of the phenomena connected with motion and stability of water films in the last stages of steam turbines is expected to allow effective measures to be taken against erosion of turbine blades. It may also help to solve many other technical problems involving thin liquid films occurring in various devices. Earlier work was mainly concerned with the simplest model of the film motion, i.e. a film driven by shear stress at the interface. In the case of a steam turbine blade channel there occur substantial variations of the cross-sectional area, resulting in strong pressure gradients. The subject of this paper is an analysis of the pressure gradient effect on the film flow and its stability. The paper is a summary of [th results obtained recently at the Department of Thermodynamics and Heat Transfer 5e÷7].

2. Gas-Phase Flow in the Channel

In the literature the liquid film flowing over the channel wall is usually considered separately from the gas-phase flow. Both phases are coupled only by continuity of shear stress at the interface. In this paper both phases are considered jointly. A possible simplification of the model is indicated in subsequent analysis.

Let us consider the channel shown in Fig. 1 with the gas phase flowing through the channel and a liquid film moving along its walls. At a given cross section $A(x)$ flow para-

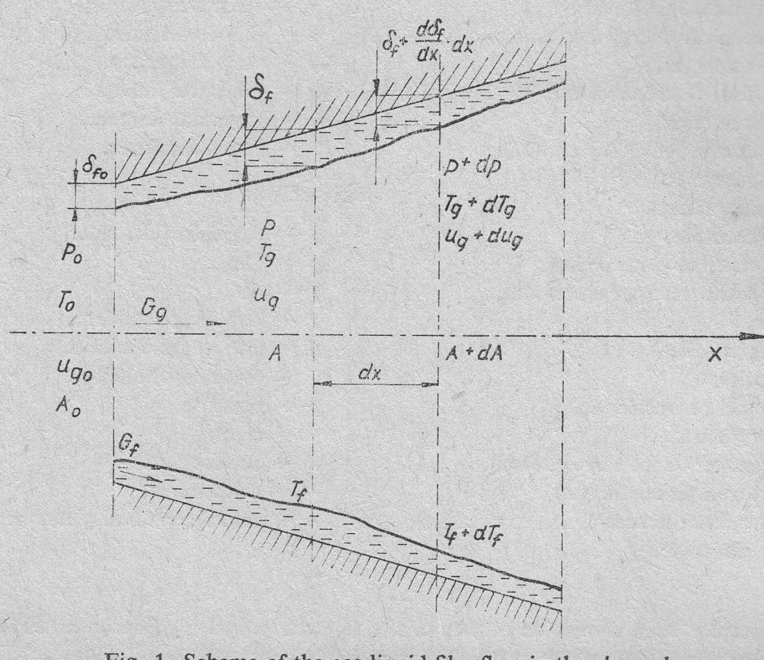

Fig. 1. Scheme of the gas-liquid film flow in the channel

meters are u_g, T_g, p_g, u_f and T_f. At $x+dx$ the area is $A+dA$. We allow momentum and heat transfer between gas and liquid film and assume further:

1) mass transfer between gas and film is negligible,
2) the flow is adiabatic,
3) energy dissipation in the gas phase is negligible,
4) flow is one-dimensional in the gas phase,
5) the gas obeys the perfect gas law,
6) film properties are constant,
7) the displacement effect of the liquid film on the gas phase is negligible,
8) film flow is laminar.

The above assumptions lead to the following equations:

– continuity equation for the gas phase

$$G_g = \rho_g u_g A, \tag{1}$$

– continuity equation for the liquid film

$$G_f = \rho_f u_f \delta_f U \tag{2}$$

– combined momentum equation

$$-dp = G_f \frac{du_f}{\delta_f U} + G_g \frac{du_g}{A}, \tag{3}$$

– combined energy equation

$$G_f\left(c_{pf} T_f + \frac{u_f^2}{2}\right) + G_g\left(c_{pg} T_g + \frac{u_g^2}{2}\right) = \text{const}. \tag{4}$$

Together with the equation of state of the gas phase

$$p = \rho_g R_g T_g \tag{5}$$

from eq. (1)÷(5) one obtains the following differential relationships

$$\frac{du_g}{dx} = \frac{\dfrac{R_g T_g}{A}\dfrac{dA}{dx} - \dfrac{G_f}{G_g}\left(-\dfrac{c_{pf}}{c_{pg}} R_g \dfrac{dT_f}{dx} - \dfrac{R_g u_f}{c_{pg}}\dfrac{du_f}{dx} + \dfrac{A}{U\delta_f} u_g \dfrac{du_f}{dx}\right)}{u_g\left(1 - \dfrac{R_g}{c_{pg}} - \dfrac{R_g T_g}{u_g^2}\right)}, \tag{6}$$

$$\frac{dT_g}{dx} = \frac{T_g}{A}\frac{dA}{dx} - \frac{G_f}{G_g}\frac{u_g}{R_g \delta_f}\frac{A}{U}\frac{du_f}{dx} - u_g \frac{du_g}{dx}\left(\frac{1}{R_g} - \frac{T_g}{u_g^2}\right), \tag{7}$$

$$\frac{dp}{dx} = G_g R_g\left(\frac{1}{A u_g}\frac{dT_g}{dx} - \frac{T_g}{A^2 u_g}\frac{dA}{dx} - \frac{T_g}{A u_g^2}\frac{du_g}{dx}\right). \tag{8}$$

The above relationships together with equations describing liquid phase motion allow the numerical computation of the two-phase gas-liquid flow when the liquid flows along wall.

In the case of $G_f = 0$, eq. (6)÷(8) reduce to the standard relations of one-dimensional gas dynamics

$$\frac{du}{dx}=\frac{\frac{T_g}{A}R_g\frac{dA}{dx}}{u_g\left(1-\frac{R_g}{c_{pg}}-\frac{R_gT_g}{u_g^2}\right)}, \tag{9}$$

$$\frac{dT_g}{dx}=\frac{T_g}{A}\frac{dA}{dx}-u_g\frac{du_g}{dx}\left(\frac{1}{R_g}-\frac{T_g}{u_g^2}\right), \tag{10}$$

$$\frac{dp}{dx}=R_gG_g\left(\frac{1}{Au_g}\frac{dT_g}{dx}-\frac{T_g}{A^2u_g}\frac{dA}{dx}-\frac{T_g}{u_g^2}\frac{du_g}{Adx}\right). \tag{11}$$

If the gas flow is further considered incompressible eq. (1) and (3) give

$$\frac{1}{\rho_g}\frac{dp}{dx}=-u_g\frac{du_g}{dx} \tag{12}$$

and

$$\frac{du_g}{dx}=-\frac{G_g}{\rho_g}\frac{1}{A^2}\frac{dA}{dx}, \tag{13}$$

while eq. (13) and (12) yield the relationship for the pressure gradient

$$\frac{1}{\rho_g}\frac{dp}{dx}=\frac{G_g^2}{\rho_g^2A^3}\frac{dA}{dx} \tag{14}$$

which suggests that the pressure changes appear only as a result of the variable channel area.

3. Smooth Film Flow

The momentum equation of a thin liquid film flowing along the channel wall may be written in a form

$$u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu_f\frac{\partial^2u_x}{\partial y^2}-\frac{1}{\rho_f}\frac{\partial p}{\partial x} \tag{15}$$

and the continuity equation is

$$u_y=-\int_0^y\frac{\partial u_x}{\partial x}dy, \tag{16}$$

because $u_y=0$ for $y=0$.

Introducing eq. (16) into eq. (15) one obtains

$$u_x\frac{\partial u_x}{\partial x}-\frac{\partial u_x}{\partial y}\int_0^y\frac{\partial u_x}{\partial x}dy=\nu_f\frac{\partial^2u_x}{\partial y^2}-\frac{1}{\rho_f}\frac{\partial p}{\partial x}. \tag{17}$$

The boundary conditions are

for $y=0$ $$u_x=0,$$

for $y=\delta_f$ $$\frac{\partial u_x}{\partial y}=\frac{\tau_i}{\mu_f},$$

for $x=0$ $$u_x=u_0, \quad \delta_f=\delta_{f0}.$$

Equation (17) and the boundary conditions may also be rewritten:

$$\nu_f \frac{d^2 u_{x(n+1)}}{dy^2}=u_{xn}\frac{\partial u_{xn}}{\partial x}-\frac{\partial u_{xn}}{\partial y}\int\limits_0^y \frac{\partial u_{xn}}{\partial x}dy+\frac{1}{\rho_f}\frac{\partial p}{\partial x}, \tag{18}$$

for $y=0$ $$u_{xn}=0,$$

for $y=\delta_f$ $$\frac{du_{xn}}{dy}=\frac{\tau_i}{\mu_f},$$

for $x=0$ $$u_x=u_0, \quad \delta_f=\delta_{f0}.$$

The form of eq. (18) allows solution through succesive approximations. The index n stands for the number of the approximation, each of which must satisfy the above boundary conditions.

Assuming the zeroth approximation $u_{x,0}=0$ one obtains the first approximation to the velocity profile within the liquid layer in the form

$$u_{x1}=\frac{1}{\mu_f}\frac{dp}{dx}\frac{y^2}{2}+\left(\frac{\tau_i}{\mu_f}-\frac{1}{\mu_f}\frac{dp}{dx}\delta_f\right)y. \tag{19}$$

Putting the mean film velocity

$$\bar{u}_x=\frac{1}{\delta_f}\int\limits_0^{\delta_f} u_x dy=\frac{1}{6\mu_f}\frac{dp}{dx}\delta_f^2+\left(\frac{\tau_i}{\mu_f}-\frac{1}{\mu_f}\frac{dp}{dx}\delta_f\right)\frac{\delta_f}{2}=\frac{\tau_i\,\delta_f}{\mu_f}\,\frac{3K-2}{6K} \tag{20}$$

into eq. (19) yields

$$u_x=\frac{3\bar{u}_x}{3K-2}\left[\left(\frac{y}{\delta_f}\right)^2+2(K-1)\left(\frac{y}{\delta_f}\right)\right], \tag{21}$$

where the parameter

$$K=\frac{\tau_i}{\frac{dp}{dx}\delta_f} \tag{22}$$

expresses the ratio of the shear forces to those due to the pressure gradient.

When the pressure gradient is negligibly small eq. (19) results in the known velocity profile of a shear driven film

$$u_{x1}\cong\frac{\tau_i}{\mu_f}y. \tag{23}$$

Further approximations of the solution of eq. (18) have a complicated analytic form and are presented in [5].

In the same paper an analytic solution has been presented for the case of a constant pressure gradient along the channel.

4. Heat Transfer between the Gas and the Liquid Film

Heat transfer between the gas and the film is by convection nature. For turbulent gas flow the heat transfer coefficient may be determined from the well-known Dittus-Boelter correlation [8]

$$Nu=0{,}023Re_g^{0,8}Pr_g^{0,4}, \tag{24}$$

where

$$Nu=\frac{\alpha d_e}{\lambda_g}, \qquad Re_g=\frac{u_g d_e}{\nu_g}$$

are respectively, the Nusselt and Reynolds numbers, based on the hydraulic channel diameter $d_e=4A/U$ and $Pr_g=\nu_g/a_g$.

The simplified energy equation for the liquid layer takes the form

$$\frac{dT_f}{dx}=\frac{\alpha U}{G_f c_{pf}}(T_f-T_g) \tag{25}$$

and with the aid of eq. (24)

$$\frac{dT_f}{dx}=0{,}023Re_g^{0,8}Pr_g^{0,4}\frac{\lambda_g U}{d_e G_f c_{pf}}(T_f-T_g). \tag{26}$$

Equation (26) together with eq. (1)÷(8) and eq. (20) determine the parameters of the two-phase liquid-vapor flow. Their numerical integration proceeds iteratively. The gas flow in a given channel is solved first assuming no liquid film on the walls. Then the liquid phase flow is determined. In turn the influence of the liquid phase flow on the gas flow is computed.

The appropriate shear stress is determined from standard, dry-wall, friction coefficient correlations and the local gas velocity.

5. Liquid Film with Small-Amplitude Waves

The influence of the interface shear stress and of the pressure gradient on the flow of a thin wavy liquid film will now be considered. The solution of such a flow has been obtained with the aid of a method proposed by Kapica [9]. Earlier discussions of the wavy thin liquid film flow concentrated mainly on a gravitational flow [9 - 12].

In order to predict the wave length of the wavy film a small-amplitude wave (Fig. 2) has been considered. The momentum equation of the unsteady film flow, taking into account

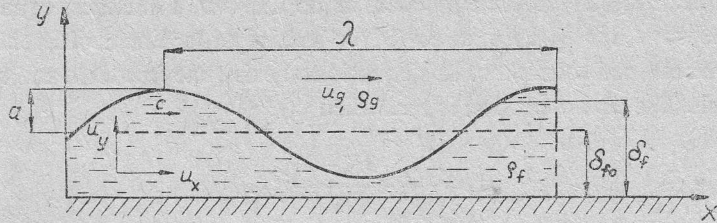

Fig. 2. Scheme of the wavy liquid film flow

the capillary forces at the interface and making use of the continuity equation is

$$\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}-\left(\int\limits_0^y\frac{\partial u_x}{\partial x}dy\right)\frac{\partial u_x}{\partial y}=\frac{\sigma}{\rho_f}\frac{d^3\delta_f}{dx^3}-\frac{1}{\rho_f}\frac{\partial p}{\partial x}+\nu_f\frac{\partial^2 u_x}{\partial y^2}. \tag{27}$$

The continuity equation yields further

$$\frac{\partial\delta_f}{\partial t}=-\frac{\partial}{\partial x}\left(\int\limits_0^{\delta_f}u_x dy\right)=-\frac{\partial(\bar{u}_x\delta_f)}{\partial x}. \tag{28}$$

The unknown velocity profile u_x is a function of the coordinates x and y and of the time t. Since the length of the waves considered is large in comparison with the film thickness δ_f eq. (27), (28) may be averaged over the film thickness δ_f.

Introducing the velocity profile eq. (21) into eq. (27), (28) and assuming the mean film velocity $\bar{u}_x=\bar{u}_x(x, t)$ one obtains after the averaging with respect to y

$$[\tfrac{1}{3}+(K-1)]\frac{\partial\bar{u}_x}{\partial t}+\frac{3\bar{u}_x}{3K-2}[\tfrac{1}{15}+\tfrac{1}{3}(K-1)+\tfrac{2}{3}(K-1)^2]\frac{\partial\bar{u}_x}{\partial x}=$$

$$=\frac{\sigma}{\rho_f}\frac{3K-2}{3}\frac{d^3\delta_f}{dx^3}+\frac{2\nu_f\bar{u}_x}{\delta_f^2}-\frac{1}{\rho_f\delta_f}\frac{3K-2}{3K}\tau_i. \tag{29}$$

It is further assumed that for the small amplitude waves the film thickness may be expressed as

$$\delta_f=\delta_{fo}+\delta_{fo}\,\varphi=\delta_{fo}(1+\varphi), \tag{30}$$

where δ_{fo} — mean thickness of the wavy film, $\delta_{fo}\,\varphi$ — deviation of the film surface from the mean thickness.

The dependence of the film thickness δ_f and of the mean velocity $\bar{u}_x$ on the argument x-ct (wave disturbance propagation) is also accounted for in the analysis.

The first approximation to the function φ has been obtained from the continuity and momentum equations in the form

$$\frac{\sigma}{\rho_f}\delta_{fo}\frac{3K-2}{3}\frac{d^3\varphi}{dx^3}-(c-u_0)\left\{\frac{3u_0}{3K-2}[\tfrac{1}{15}+\tfrac{1}{3}(K-1)+\tfrac{2}{3}(K-1)^2]-c[\tfrac{1}{3}+(K-1)]\right\}\frac{d\varphi}{dx}+$$

$$+\frac{2\nu_f}{\delta_{fo}^2}(c-3u_0)\,\varphi+\frac{2\nu_f u_0}{\delta_{fo}^2}-\frac{1}{\rho_f\delta_f}\frac{3K-2}{3K}\tau_i=0, \tag{31}$$

where u_0 is the mean velocity of the equivalent smooth film of a thickness equal to the mean thickness of the wavy film δ_{f0}. Equation (31) is a third order linear differential equation. For a periodic undamped solution of this equation to exist, the free term and the coefficient of φ must vanish. The first condition is met when

$$u_0 = \frac{\tau_i}{\mu_f \delta_f} \frac{3K-2}{6K} \delta_{f0}^2 . \tag{32}$$

Comparing eq. (32) with the mean velocity of a non-wavy film (20) one obtains

$$u_0 = \bar{u}_x . \tag{33}$$

It follows that in the first approximation the mean thickness of a wavy film δ_{f0} is equal to that of the smooth film δ_f. The second condition implies that the wave propagation velocity

$$c = 3u_0 . \tag{34}$$

Taking into account the above conditions and the relationships (33, 34) the function φ is obtained in the form

$$\varphi = a \sin(kx - \omega t), \tag{35}$$

where k is the wave number given by

$$k = \left[\frac{6}{5} \frac{35K^2 - 45K + 14}{(3K-2)^2} \frac{u_0^2 \rho_f}{\sigma \delta_{f0}} \right]^{\frac{1}{2}} . \tag{36}$$

When the parameter K described by eq. (22) tends to infinity, which corresponds to the shear driven film flow, the wave number k is given by

$$k = \sqrt{\frac{4{,}66 u_0^2 \rho_f}{\sigma \delta_{f0}}} \tag{37}$$

and is slightly larger than for the gravitational film. The constant in eq. (37) equals 4.2 for a gravitational film and takes the value of 4.66 in the case under consideration.

Relationship (36) may also be written

$$k = \left[\frac{f(K) u_0^2 \rho_f}{\sigma \delta_{f0}} \right]^{\frac{1}{2}} \tag{38}$$

Some values of the function $f(K)$ are listed in the Table 1.

Table 1

K	0	10	100	1000	10 000
$f(K)$	4.2	4.6898	4.6689	4.6669	4.6667

The wave number predicted from eq. (38) allows one to calculate the wave length

$$\lambda = \frac{2\pi}{k} . \tag{39}$$

It follows from the Table 1 that moderate pressure gradients affect the wave number and consequently the wave length only slightly. In technical calculations the influence of the pressure gradient on the wavy film flow may be neglected.

6. Effect of the Pressure Gradient on the Film Breakdown into Rivulets

References [1, 2, 4, 13] consider the breakdown of shear driven films into rivulets. For that case the velocity profile within the liquid is given by eq. (23).

Let us consider the film breakdown under substantial pressure gradients, when the film velocity profile is described by eq. (19) or (21). The mechanical energy of the system shown in Fig. 3 consists of the kinetic energy and the surface energy of all the physical surfaces. The total mechanical energy per unit width of the film before breakdown is given by

$$e_f=\frac{E_f}{s}=\frac{\rho_f}{2}\int\limits_0^{\delta_f} u_f^2(z)\,dy+\sigma_{sf}+\sigma_{fg}=\frac{\rho_f}{2}\frac{1}{\mu_f^2}\left(\frac{\partial p}{\partial x}\right)^2[\tfrac{2}{15}-\tfrac{5}{12}K+\tfrac{1}{3}K^2]\,\delta_f^5+\sigma_{sf}+\sigma_{fg}. \tag{40}$$

The mass flow rate in the film per unit width before breakdown is

$$q_f=\frac{G_f}{s}=\rho_f\int\limits_0^{\delta_f} u_f(y)\,dy=\frac{\rho_f}{\mu_f}\frac{\partial p}{\partial x}[\tfrac{1}{2}K-\tfrac{1}{3}]\,\delta_f^3. \tag{41}$$

Assuming the surface tension σ_{fg} to be constant and neglecting gravity effects the rivulet forms a sector of a circle as shown in Fig. 3. The sector is determined by the central angle $2\,\theta_0$, where θ_0 is the contact angle.

The velocity profile in a given cross section of the rivulet of thickness $\delta_r(z)$ is assumed identical to that in a film of constant thickness $\delta_r(z)$. This assumption was justified in [13]. In general, the rivulet velocity profile may be written as a function $u_r(z, y)$. Hence the

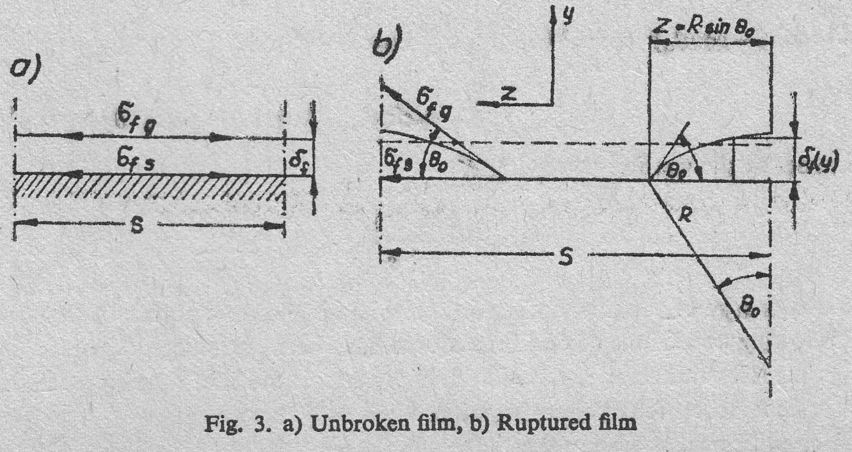

Fig. 3. a) Unbroken film, b) Ruptured film

rivulet mass flow rate per unit width of the film is

$$q_r=\frac{G_r}{s}=\frac{2\rho_f}{s}\int_0^{z_1}\int_0^{\delta_r(z)} u_r(z,y)\,dz\,dy=\frac{2\rho_f}{\mu_f}\frac{\partial p}{\partial x}R^4\left[\tfrac{1}{2}K\left(\frac{\delta_f}{R}\right)I_2-\tfrac{1}{3}I_3\right], \tag{42}$$

where

$$I_2=\int_0^{\theta_0}(\cos\theta-\cos\theta_0)^2\cos\theta\,d\theta=\sin\theta_0-\tfrac{1}{3}\sin^3\theta_0-\theta_0\cos\theta_0$$

$$I_3=\int_0^{\theta_0}(\cos\theta-\cos\theta_0)^3\cos\theta\,d\theta=-\tfrac{1}{4}\cos^3\theta_0\sin\theta_0-\tfrac{13}{8}\cos\theta_0\sin\theta_0-\tfrac{3}{2}\theta_0\sin^2\theta_0+\tfrac{15}{8}\theta_0.$$

The total mechanical energy per unit width of the film is given by

$$e_r=\frac{E_r}{s}=\frac{\rho_f}{2s}\int_0^{z_1}\int_0^{\delta_r(z)} u_r^2(z,y)\,dz\,dy+\frac{1}{s}\left[2R\sigma_{fs}\sin\theta_0+2R\theta_0\sigma_{fg}+(s-2R\sin\theta_0)\sigma_{sg}\right]=$$

$$=\frac{\rho_f}{2s}\int_0^{z_1}\int_0^{\delta_r(z)} u_r^2(z,y)\,dz\,dy+\frac{1}{s}(2R\theta_0+\cos\theta_0-R\sin 2\theta_0)\sigma_{fg}+\sigma_{fs}= \tag{43}$$

$$=\frac{\rho_f}{\mu_f^2}\left(\frac{\partial p}{\partial x}\right)^2R^6\left[\tfrac{2}{15}I_5-\tfrac{5}{12}K\left(\frac{\delta_f}{R}\right)I_4+\tfrac{1}{3}K^2\right]+\frac{1}{s}(2R\theta_0+\cos\theta_0-R\sin 2\theta_0)\sigma_{fg}+\sigma_{fs},$$

where

$$I_4=\int_0^{\theta_0}(\cos\theta-\cos\theta_0)^4\cos\theta\,d\theta=-2\cos^3\theta_0(\tfrac{1}{5}\sin 2\theta_0+\theta_0)+(\sin\theta_0-\tfrac{1}{3}\sin^3\theta_0)\times$$
$$\times(\tfrac{4}{5}+6\cos^2\theta_0)-4\cos\theta_0(\tfrac{3}{8}\theta_0+\tfrac{1}{4}\sin 2\theta_0+\tfrac{1}{32}\sin 4\theta_0),$$

$$I_5=\int_0^{\theta_0}(\cos\theta-\cos\theta_0)^5\cos\theta\,d\theta=\theta_0(\tfrac{5}{16}+\tfrac{15}{4}\cos^2\theta_0+\tfrac{5}{12}\cos^4\theta_0)+$$
$$-\sin\theta_0(\tfrac{113}{48}\cos\theta_0+\tfrac{97}{24}\cos^3\theta_0+\tfrac{1}{6}\cos^5\theta_0).$$

Let us introduce the wetting parameter X

$$X=\frac{2R\sin\theta_0}{s}. \tag{44}$$

The total energy of the film broken into rivulets is then a function of that parameter. The following conditions of film breakdown have been formulated:
- film mass flow rate and mechanical energy per unit width of the film remain constant,
- new flow structure (rivulets) is stable.

These conditions may be expressed as

$$\frac{G_f}{s}=\frac{G_r}{s}, \tag{45}$$

$$e_f=e_r, \tag{46}$$

$$\frac{de_r}{dX}=0 \qquad \frac{\partial^2 e_r}{\partial X^2}>0 \tag{47}$$

for $X\leqslant 1$.

From the energy balance eq. (46) there results

$$(\delta_f^+)^5=\frac{X\left(\dfrac{\theta_0}{\sin\theta_0}-\cos\theta_0\right)-(1-\cos\theta_0)}{\frac{2}{15}-\frac{5}{12}K+\frac{1}{3}K^2-\dfrac{X}{\left(\dfrac{\delta_f}{R}\right)^5\sin\theta_0}\left[\frac{2}{15}I_5-\frac{5}{12}K\left(\dfrac{\delta_f}{R}\right)I_4+\frac{1}{3}K^2\left(\dfrac{\delta_f}{R}\right)^2 I_3\right]}, \tag{48}$$

where

$$\delta_f^+=\left[\frac{\rho_f}{2\mu_f^2\sigma_{f,g}}\left(\frac{\partial p}{\partial x}\right)^2\right]^{1/5}\delta_f .$$

On the other hand from eq. (47)

$$\frac{(\delta_f^+)^5}{\sin\theta_0\left(\dfrac{\delta_f}{R}\right)^6}\left\{\frac{2}{15}I_5\left(\frac{\delta_f}{R}\right)-\frac{5}{12}K\left(\frac{\delta_f}{R}\right)^2 I_4+\frac{1}{3}K^2\left(\frac{\delta_f}{R}\right)^3 I_3+X\frac{d\left(\dfrac{\delta_f}{R}\right)}{dX}\left[\frac{5}{3}K\left(\frac{\delta_f}{R}\right)I_4+\right.\right.$$

$$\left.\left.-\frac{2}{3}I_5-K^2 I_3\left(\frac{\delta_f}{R}\right)^2\right]\right\}+\left(\frac{\theta_0}{\sin\theta_0}-\cos\theta_0\right)=0 \tag{49}$$

and from eq. (45)

$$\frac{X}{\sin\theta_0}=\frac{\left(\dfrac{\delta_f}{R}\right)^3\left(\frac{2}{3}-K\right)}{\frac{2}{3}I_3-KI_2\left(\dfrac{\delta_f}{R}\right)}. \tag{50}$$

Putting eq. (50) into eq. (48) one obtains an equation for the dimensionless film thickness at the moment of breakdown

$$(\delta_f^+)^5=$$

$$=\frac{\left(\dfrac{\delta_f}{R}\right)^3\left(\frac{2}{3}-K\right)\left(\theta_0-\frac{1}{2}\sin 2\theta_0\right)-(1-\cos\theta_0)\left[\frac{2}{3}I_3-KI_2\left(\dfrac{\delta_f}{R}\right)\right]}{\left(\frac{2}{15}-\frac{5}{12}K+\frac{1}{3}K^2\right)\left[\frac{2}{3}I_3-KI_2\left(\dfrac{\delta_f}{R}\right)\right]-\left(\dfrac{\delta_f}{R}\right)^{-2}\left(\frac{2}{3}-K\right)\left[\frac{2}{15}I_5-\frac{5}{12}K\left(\dfrac{\delta_f}{R}\right)I_4+\frac{1}{3}K^2\left(\dfrac{\delta_f}{R}\right)^2 I_3\right]}. \tag{51}$$

Another relationship results from eq. (49) and eq. (50)

$$(\delta_f^+)^5 = \frac{-\left(\frac{\delta_f}{R}\right)^5 (\theta_0 - \frac{1}{2}\sin 2\theta_0)\left[I_3 - KI_2\left(\frac{\delta_f}{R}\right)\right]}{-\frac{4}{45}I_3I_5 + \left(\frac{\delta_f}{R}\right)(\frac{1}{5}I_2I_5K + \frac{5}{36}KI_3I_4) - \frac{5}{12}K^2\left(\frac{\delta_f}{R}\right)^2 I_2I_4 + \frac{1}{6}K^3\left(\frac{\delta_f}{R}\right)^3 I_2I_3}. \quad (52)$$

Equations (51) - (52) form a set of two equations with the unknowns δ_f^+ and (δ_f/R). This allows the determination of the dimensionless film thickness at breakdown δ_f^+ as a function of the parameter K and the contact angle θ_0. Calculations of δ_f^+ are presented in Fig. 4. A special geometrical procedure is necessary to find the value of δ_f from Fig. 4, since δ_f occurs in both δ_f^+ and K.

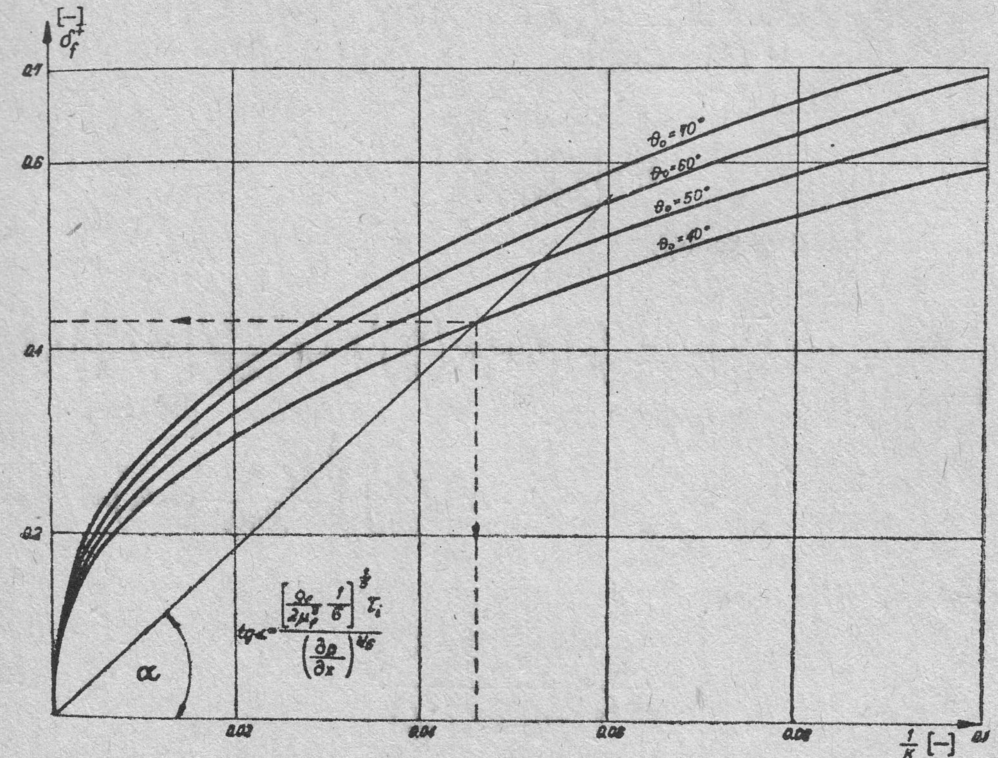

Fig. 4. Dimensionless thickness of the broken film δ_f^+ versus $1/K$ parameter for different values of the contact angle

It follows from the definitions of K and δ_f^+ that δ_f is directly proportional to δ_f^+ and to the reciprocal of K. For given physical properties, pressure gradient and shear stress at the interface it is possible to calculate the factors of proportionality. In turn, one can draw a line from the origin (δ_f^+, $1/K$) and crossing the curve corresponding to a given contact angle θ_0.

For that point one finds the values of δ_f^+ and K and, in consequence, the required film thickness δ_f.

7. Conclusions

Relationships describing gas phase and liquid phase flow in a channel of variable area are presented in this paper.

Variable channel area affects mainly the gas phase parameters resulting in variable pressure gradients along the channel axis and in variable shear stress at the interface. Variable pressure gradients and shear stress at the interface drive the liquid phase assumed to be incompressible. Wavy film flow and its breakdown into rivulets have also been considered. The resulting formulas allow an analytic solution under substantial simplifications only.

In more general cases numerical techniques are necessary. Therefore, general conclusions are not available. However, the influence of the pressure gradient on film flow in channels of technically important geometries can be studied. Detailed assumption concerning the channel profile are then necessary.

The analysis of the wavy liquid film reveals only a slight influence of the pressure gradient on the wave motion. This influence may be neglected in engineering calculations. Numerical calculations of film breakdown also indicate only a slight influence of the pressure gradient on the film breakdown process.

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Analiza wpływu gradientu ciśnienia na przepływ filmu i jego stabilność

Streszczenie

W pracy podano zależności opisujące ruch fazy gazowej i fazy ciekłej w kanale o zmiennym przekroju. Zmiana przekroju kanału wpływa głównie na parametry fazy gazowej, co prowadzi do zmiennych gradientów ciśnienia wzdłuż osi kanału oraz zmiennych naprężeń stycznych na granicy rozdziału między fazą gazową a ciekłą. Zmienne gradienty ciśnienia oraz naprężenia styczne na granicy rozdziału faz powodują ruch fazy ciekłej, którą traktowano jako nieściśliwą. Rozważono ruch falowy oraz pękanie filmu cieczowego na strugi. Opracowane zależności tylko przy dużych uproszczeniach prowadzą do rozwiązań analitycznych. W przypadkach bardziej ogólnych wymagają zastosowania numerycznych technik obliczeniowych. Analiza ruchu falowego filmu cieczowego płynącego po ściance kanału wykazuje, że wpływ gradientu ciśnienia na ruch falowy jest niewielki i w obliczeniach inżynierskich może być nie uwzględniany. Wpływ gradientu ciśnienia na proces pękania filmu wydaje się również niewielki.

Анализ влияния градиента давления на течение пленки жидкости и ее устойчивость

Резюме

В работе представлены зависимости описывающие движение газовой и жидкой фаз в канале переменного сечения. Изменение поперечного сечения канала влияет в основном на параметры газовой фазы, что приводит к переменным градиентам давления вдоль оси канала, а также к переменным касательным натяжениям на границе раздела газовой и жидкой фаз. Переменные градиенты давления и касательные натяжения на границе раздела фаз вызывают движение жидкой фазы, которую принято считать несжимаемой. Рассматривается волновое движение и разрыв жидкостной пленки на струи. Выведенные зависимости только при больших упрощениях ведут к аналитическим решениям. В более общих случаях необходимо применять численную расчетную технику. Анализ волнового движения жидкостной пленки, текущей по стенке канала доказывает, что влияние градиента давления на волновое движение невелико и в инженерских расчетах пренебрежимо. Влияние градиента давления на процесс разрыва пленки кажется быть также незначительным.