

VARIATIONAL AND EXTREMUM PRINCIPLES IN MACROSCOPIC SYSTEMS

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Chapter 8

AN INTRODUCTION TO VARIATIONAL DERIVATION OF THE PSEUDOMOMENTUM CONSERVATION IN THERMOHYDRODYNAMICS

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Abstract

In this study we will present a reconstruction and revaluation of a few basic, pioneering variational principles in thermohydrodynamics. In particular, most of our attention will be aimed at Gibbs', Natanson's and Eckart's principles. All of them either contain preliminary forms of pseudomomentum and pseudomomentum flux notions or are natural ground for them. They also present a method to obtain pseudomomentum balance equations from properly constructed variational principles. The study is completed with analysis of other variational principles and variational grounds for a 'Generalized Lagrangian Mean' (GLM) approach recently applied in atmospheric and ocean turbulence description.

Keywords: pseudomomentum; pseudomomentum flux; the chemical-potential tensor; Lagrangian variational principles; The Gibbs; Natanson; Eckart principles; material vorticity; configurational forces; acoustic pseudomomentum; wave momentum; Generalized Lagrangian Mean

1. Introduction

Variational calculus methods are widely applied throughout various branches of physics. We should stress that their applications fairly exceed their use as a tool for solving particular problems (no matter how important the problems may be). 'Variational principles' reflect in fact the most general physical laws of various fields of physics ranging from classical mechanics to elementary particles theory.

The variational description has numerous advantages. First, we work with a functional—a notion that is simpler and closer to physical experience than a derivative. Functional properties result in its indifference of the coordinate system and enable the use of approximate methods to find a solution. Variational physical interpretations are especially useful when analytical solutions of differential equations for mass, momentum and energy balance are nonexistent or present significant difficulties.

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The variational approach changes our traditional opinion that a fluid continuum is naturally described in the Eulerian (or spatial) coordinates and a solid continuum is described in the Lagrangian (material) ones. Often, a physical system consists of a set of fields some of which are naturally accounted for the Eulerian description while the others are more easily expressed in the Lagrangian description. In the Eulerian description we fix the observer's reference frame in a point of space and observe characteristic parameters of a fluid medium flowing through the point. In the Lagrangian description, in turn, we fix an observer's reference frame within a material particle and we follow it looking for an actual position as a function of beginning coordinates and time.

Traditionally, the set of governing equations for thermohydrodynamics (mass, momentum and energy balances) has been formulated in the Eulerian description as a basic postulate of the Newtonian mechanics and the Joulean thermodynamics. However, starting from a variational principles' point of view, a set of conservative balances simultaneously with the Euler–Lagrange 'equations of motion' can be obtained. This grows from a generalized form of Noether's invariance theorem [1,2]. In particular, when the total Lagrangian is invariant with respect to time and space transformation, besides of the variational 'equation of motion', we can obtain the conservation of a canonical energy–momentum tensor in a form of four following equations: $T_{\alpha\beta,\beta} = 0$, $\alpha, \beta = t, z, y, z$ [3–7].

This quite general conservation theorem, arising in the different branches of the classical field theory, can be applied to the variational formulation of fluid thermodynamics. This will lead to conservation of the canonical energy–momentum flux that has particular names of the conservation of pseudomomentum and conservation of pseudoenergy. This set of additional equations is obtainable in a relatively easy way if one starts from a variational principle formulated in the Lagrangian description. However, such an approach results in some difficulties in physical interpretations, especially if someone seeks simple connections and equivalence between the Euler–Lagrange 'equation of motion' in the Eulerian coordinates and the Euler–Lagrange 'equation of motion' in the Lagrangian coordinates [8–12].

Variational methods also include analysis utilizing minimum-energy criterion (with constant entropy) and minimum-entropy production. Unfortunately, variational techniques in thermodynamics of discontinuities (of shock wave, phase transitions, linear defects' (eddies) type) are scarcely used in mathematical model investigations, despite the fact that they originate from Gibbs' days. Nonetheless, Gibbs' approach is still valid, as the literature shows [13].

In our study we will present a reconstruction and revaluation of a few basic, pioneering variational principles in thermohydrodynamics. In particular, most of our attention will be aimed at Gibbs' [14], Natanson's [15] and Eckart's [16] principles. All of them either contain preliminary forms of pseudomomentum and pseudomomentum flux notions or are natural ground for them. They also introduce a method to obtain pseudomomentum balance equations from properly constructed variational principles both in the Eulerian and the Lagrangian representations. The study is completed with analysis of other variational principles and variational grounds for the 'Generalized Lagrangian Mean' (GLM) approach recently applied in atmospheric and ocean turbulence description [17–19].

2. Notion of pseudomomentum vector and pseudomomentum flux tensor

The definition of the pseudomomentum and its conservation is strictly connected with the material Lagrangian L^m usually defined as a sum of kinetic T , internal ε and potential Ω energy, which is completed with the work of several constraints like, for instance, an adiabatic process:

$$L^m = \rho_0(T - \varepsilon - \Omega) - \theta(\eta, \tau) = L^m(x_{i,\tau}, x_{i,A}, \pi, X_A, \tau), \quad (1)$$

which depends on a set of unknowns fields x_i , η , its derivatives $x_{i,A}$, $x_{i,\tau}$ and independent variables X_A , τ . Throughout this study we use Cartesian tensor notation; adhering to the current literature we denote Eulerian coordinates of \vec{x} , \vec{v} , ∇ by lower-case indices ($i, j, k = x, y, a$) and Lagrangian components of \vec{X} by capital indices ($A, B = X, Y, Z$). A comma followed by an index denotes partial differentiation with respect to a coordinate or time. Then, taking into account that L^m is a function only first derivatives $x_{i,A}$, $x_{i,\tau}$ of unknown functions, from the Noether theorem applied for the variation of Lagrangian coordinates δX_A , a general pseudomomentum conservation equation is obtained:

$$\frac{\partial}{\partial \tau} P_A + \frac{\partial}{\partial X_B} P_{AB} = F_A^{\text{inh}} \quad \text{or} \quad \partial_\tau \vec{P} + \text{DIV}(\vec{P}) = \vec{F}^{\text{inh}}. \quad (2)$$

The pseudomomentum vector $P_A = \rho_0 u_A$ (in m/s per unit reference volume), has been defined using specific pseudomomentum u_A (m/s per unit mass) [16,20,21]. The above quantity is strictly related with a structure of Lagrangian (1) and is defined to be ([16], Eq. (23)):

$$P_A = x_{i,A} \frac{\partial L^m}{\partial x_{i,A}} = x_{i,A} \frac{\partial(\rho_0 T)}{\partial x_{i,\tau}}, \quad u_A = x_{i,A} \frac{\partial(T)}{\partial x_{i,\tau}} = x_{i,A} x_{i,\tau}. \quad (3)$$

The last form of the specific pseudomomentum u_A follows from the classical form of kinetic energy (per unit mass) $T = 1/2(\vec{v} \cdot \vec{v}) = 1/2(x_{i,\tau} x_{i,\tau})$. Components of this vector have been denoted by Eckart as: $u_\alpha, u_\beta, u_\gamma$, respectively.

Next, the flux of pseudomomentum is defined to be:

$$P_{AB} = -L^m \delta_{AB} + x_{i,A} \frac{\partial L^m}{\partial x_{i,B}}. \quad (4)$$

In analogy with the Cauchy flux of momentum, this tensor is sometimes called the Eshelby flux tensor [22–24]. Further specification of pseudomomentum flux follows from the detailed form of Lagrangian (1), and in the case of a thermoelastic solid is [14,25]:

$$P_{AB} = \rho_0(\varepsilon + \Omega - T - \theta\eta)\delta_{AB} - F_{iA}T_{iB}, \quad (5)$$

or, in the case of elastic fluid [15,26]

$$P_{AB} = \rho_0\left(\varepsilon + \Omega + \frac{p}{\rho} - T - \theta\eta\right)\delta_{AB} = \rho_0\mu\delta_{AB}. \quad (6)$$

In the above formulae several known objects have appeared: ρ_0 is the reference density of medium, ε the specific internal energy, T the specific kinetic energy, $\Omega(x_i)$ the specific gravitational potential, θ the thermodynamical temperature, η the specific entropy

(in Gibbs' notation), μ the chemical-potential scalar, $F_{iA} = x_{i,A}$ the deformation gradient, $T_{iA} = \rho_0 \partial \varepsilon / \partial x_{i,A}$ the first Piola–Kirchhoff stress tensor.

The pseudomomentum flux tensor $\vec{P} = P_{AB} \vec{E}_A \otimes \vec{E}_B$ is usually addressed with a few different names; in the literature concerning nonlinear elastic solids it is called the Eshelby tensor, or material momentum flux. In thermodynamics of multiphase mixtures, the tensor \vec{P} and its Eulerian representation $\vec{p} = p_{Ai} \vec{E}_A \otimes \vec{e}_i$ are called the chemical-potential tensor [27–29]. In the Maxwell electrodynamics, an analog of pseudomomentum flux is known as an ethereal stress tensor [30–35]. In the acoustics of moving media P_{AB} is called the radiation stress or radiation pressure tensor [36–41]. In the fluid wave mechanics it is called simply the pseudomomentum flux, the material momentum flux or the wave momentum flux [21, 39, 42–47].

A simple, consistent explanation of the role of pseudomomentum conservation is probably impossible. In order to understand the origin of pseudomomentum let us observe that our elastic fluid may be subjected to two quite distinct invariance operations, either of which might be called a 'translation'. The first one is a translation of the whole medium (together with any inhomogeneities, disturbances, surface discontinuities, vortices, cavity, wave fronts) in a given space direction. If this operation leaves the action Lagrangian invariant—provided the space is homogeneous—the conserved associate quantity is physical (Newtonian) momentum.

The second translation leaves the medium itself fixed, but translates only the inhomogeneity (line, surface, volume) through the material in a given space direction. This operation is an invariant symmetry only if the physical space and fluid continuum are homogeneous. This statement leads us to the practical remark that any variation of disturbance of homogeneous material leads to the appearance of the reaction in a form of the configurational force:

$$F_A^{\text{inh}} = - \left(\frac{\partial L^m}{\partial X_A} \right)_{\text{exp}}. \quad (7)$$

In the case when 'material inhomogeneity' moves through the fluid the F_A^{inh} is interpreted as a material force or configurational force acting on it [48]. This notion is a generalization of special ones as: force on a elastic singularity [25], driving force acting on an interface [22] or bubble drift [49].

3. The Thomson–Tait variational principle

Let us consider the simplest inhomogeneity of the fluid medium such as a rigid body Σ immersed in the ideal fluid. The system with the rigid-body motion energy and kinetic energy of fluid is invariant under the following transformations: the whole fluid displacement and the displacement of the immersed rigid body by the same amount. The latter transformation—leading to the pseudomomentum conservation—consists also of the displacement of all physical properties related to the one point of the fluid to an adjacent one, without moving the medium. Therefore, in the case of the energy transfer from one body to another through the inhomogeneity surface, the variational principle should take into account the virtual displacement δN of the body surface $\partial \Sigma$.

The problem has been stated by Thomson and Tait [50] as:

$$\int_{t_1}^{t_2} dt \int_{\partial \Sigma} p \delta N dA = \int_{t_1}^{t_2} dt \delta L^m \quad (8)$$

where p is the pressure on the body surface $\partial \Sigma$, $\delta N = \delta \vec{u} \cdot \vec{N}$ the virtual displacement that acts in the normal direction to the surface. The integral over $\partial \Sigma$ represents virtual work of pressure forces on the displacement δN . The Lagrangian $L^m = \rho_0(T - \Omega)$ represents the difference of the kinetic and potential energies for fluid and solid. After numerous developing works, for instance Kirchhoff [51], Peierlis has verified [40] that the sum of the pseudomomentum of the fluid and the momentum of the rigid body is conserved. This follows from the remark that in the case of the rigid-body motion in unbounded fluid the integral of the right-hand side term (8) excludes the volume of the body, so that boundary terms vanish, which amount to minus, the net force Q_m of the body [52,53]:

$$\sum_{m=1}^6 Q_m q_m = \oint p \delta N dA. \quad (9)$$

4. Gibbs' principle extended formulation (1877)

Phase equilibrium conditions originally given by Gibbs [14] as valid for heterogeneous (with interphase surfaces) as well as for homogeneous (no interphase surfaces) phase transitions, are nowadays applied only to homogeneous (equilibrium) phase transitions. The homogenous parameters distribution assumption results in space independence of temperature (θ), pressure and chemical potential (μ). However, as the phases retain their properties and influence each other on the interphase surface (in a way specified in constitutive equations), Gibbs introduced the interphase surface (Fig. 1), where phase transitions conditions apply:

- thermal equilibrium: $\theta' = \theta''$,
- mechanical equilibrium: $p' = p''$,
- chemical equilibrium: $\mu' = \mu''$.

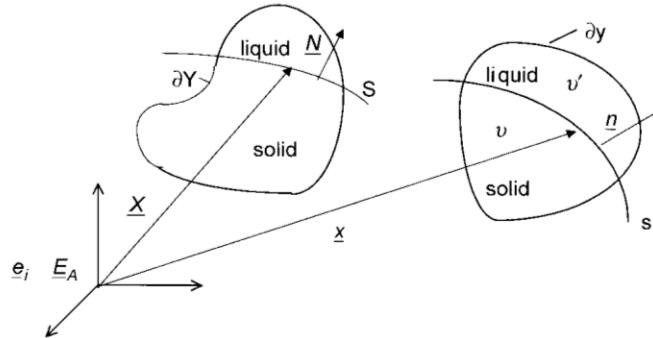


Fig. 1. Geometry of the interphase surfaces.

It turns out that the above conditions on phase equilibrium between fluid and its vapor can follow from a properly constructed variational principle. Gibbs' principle defines not only conditions for an equilibrium phase transition in fluids but also in solids, providing their deformability. The location of a point on the interphase surface S within the rigid body in the nondeformed state can be defined as:

$$\vec{X} = X\vec{E}_X + Y\vec{E}_Y + Z\vec{E}_Z = X_A\vec{E}_A, \quad (10)$$

whereas for the deformed state:

$$\vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = x_i\vec{e}_i. \quad (11)$$

Conditions for fluid and body equilibrium can be obtained if variation of the whole system (with constant entropy) is equalized to zero:

$$\begin{aligned} & \iiint_{V_0} \left[\rho_0 \theta \delta \eta + T_{iA} \delta F_{iA} + \rho_0 b_i \delta x_i \right] dV_0 + \iint_S \rho_0 \varepsilon \delta N' dS \\ & + \iiint_V \left[\theta \delta(D\eta) - p' \delta(Dv_0) + \sum_k \mu_k \delta(Dm_k) \right] dV = 0, \end{aligned} \quad (12)$$

where the deformation gradient expressing the rate of deformation is defined as ([14], Eq. (354)):

$$\vec{F} = \text{GRAD} \vec{x} = x_{iA} \vec{e}_i \otimes \vec{E}_A = F_{iA} \vec{e}_i \otimes \vec{E}_A, \quad J = \det \vec{F} = \rho_0 / \rho, \quad (13)$$

whereas the two-point, first Piola–Kirchhoff's tensor T_{iA} describing purely reversible properties of the internal energy is given in the form of ([14], Eq. (355)):

$$T_{iA} = \frac{\partial \varepsilon_0}{\partial F_{iA}} = \rho_0 \frac{\partial \varepsilon}{\partial F_{iA}} \quad \text{or} \quad \vec{T} = \rho_0 \frac{\partial \varepsilon}{\partial \vec{F}}, \quad (14)$$

$\varepsilon, \rho, \theta, \eta$ in Eq. (12) denote internal energy, density, specific temperature and specific entropy, respectively. A 'prime' sign in Eq. (12) denotes that the normal of the surface $\delta N' = \delta \vec{X}_S \cdot \vec{N}$ is directed inside the fluid.

Due to constant temperature and entropy in the whole domain and the Gauss theorem, the total energy variation can be written as ([14], Eq. (369)):

$$\begin{aligned} & - \iiint_{V_0} \frac{\partial}{\partial X_A} (T_{iA}) \delta x_i dV_0 + \iiint_{V_0} \rho_0 b_i \delta x_i dV_0 + \iint_S N_A T_{iA} \delta x_i dS \\ & + \iint_S p' \vec{n} \cdot \delta \vec{x}_S dS + \iint_S \rho_0 (\varepsilon - \theta \eta) \delta N' dS + \iint_S \left[p' J^{-1} + \sum_k \mu'_k \rho'_{0k} \right] \delta N' dS = 0. \end{aligned} \quad (15)$$

Consequently, we try to extend the principle (15). We assume that the interphase surface of a two-phase flow is being formed and changed in time. In consequence, not only basic, fixed fields are varied, but also the location of the numerical region boundary, so that $X_A \rightarrow Y_A$, $\delta X_A = Y_A - X_A$. This approach is known in variational principles theory as the Noether variational approach, which leads to the comparison of the conservation laws with the Euler–Lagrange equations for the functional extreme.

Thus, let $Y_A = Y_A(X_B, \tau)$ describe the movement of boundary and the control volume itself. Each surface in X_A is subdued to small movements described by parameter τ . This can be expressed analogically in Eulerian frame as: $y_i = y_i(x_j, \tau)$. This can be interpreted as a minimal shift of volume Y into the minimally shifted region y . In consequence we can write:

$$\delta x_i = \frac{d}{d\tau} (y_i(Y_A, \tau), \tau) \Big|_{\tau=0},$$

with boundary condition:

$$\begin{cases} x_i = y_i & (\tau = 0) \\ X_A = Y_A & (\tau = 0). \end{cases}$$

Variation of deformation can be expressed as a difference between variations of the base function and the boundary

$$\delta \left(\frac{\partial y_i}{\partial Y_A} \right) = \delta \left(\frac{\partial y_i}{\partial X_A} \right) - \frac{\partial x_i}{\partial X_B} \delta \left(\frac{\partial Y_B}{\partial X_A} \right).$$

Considering the fact that the internal energy of the solid body is a function of deformation gradient and entropy, whereas the internal energy of a fluid is a function of the deformation gradient determinant and entropy,

$$\varepsilon_s = \varepsilon_s(F_{iA}, \eta) \quad \text{and} \quad \varepsilon_l = \varepsilon_l(J, \eta),$$

on a fixed boundary we have:

$$\begin{aligned} & \delta \int \int \int_{\bar{v}_0} \rho_0 \varepsilon \left(\frac{\partial y_i}{\partial Y_A}, \eta \right) \det \left(\frac{\partial Y}{\partial X} \right) d\bar{v}_0 \\ &= \int \int \int_{v_0} \left\{ \rho_0 \frac{\partial \varepsilon}{\partial \left(\frac{\partial y_i}{\partial Y_B} \right) \Big|_{\tau=0}} \delta \left(\frac{\partial y_i}{\partial Y_B} \right) + \rho_0 \frac{\partial \varepsilon}{\partial \eta} \delta \eta + \rho_0 \varepsilon \frac{\partial}{\partial \eta} (\delta Y_A) \right\} dv_0 \\ &= \int \int \int_{v_0} \left\{ \Pi_{iA} \left[\delta \left(\frac{\partial y_i}{\partial X_A} \right) - \frac{\partial x_i}{\partial X_B} \delta \left(\frac{\partial Y_B}{\partial X_A} \right) \right] + \rho_0 \frac{\partial \varepsilon}{\partial \eta} \delta \eta + \rho_0 \varepsilon \frac{\partial}{\partial X_A} (\delta Y_A) \right\} dv_0 \\ &= \int \int \int_{v_0} \left\{ \left(-\frac{\partial}{\partial X_A} T_{iA} \right) \delta y_i - \left[\frac{\partial}{\partial X_A} \left(\rho_0 \varepsilon \delta_{AB} - \frac{\partial x_i}{\partial X_B} T_{iA} \right) \delta Y_B + \rho_0 \frac{\partial \varepsilon}{\partial \eta} \delta \eta \right] \right\} dv_0 \\ &+ \int \int_S \left\{ \delta y_i T_{iA} + \delta Y_B (\rho_0 \varepsilon \delta_{AB} - F_{iB} T_{iA}) \right\} N_A dS + \text{com. on } \partial V_0, \end{aligned} \quad (16)$$

where δ_{AB} is the Kroenecker delta.

Using Lagrange's multiplier Λ , the functional can be completed with constant entropy constraint:

$$\begin{aligned}
 \delta \left[\Lambda \int \int \int_{V_0} \rho_0 \eta \, dV_0 \right] &= \delta \left[\Lambda \int \int \int_{\tilde{V}_0} \rho_0 \eta(y_i, Y_A) \det \left(\frac{Y}{X} \right) d\tilde{V}_0 \right] \\
 &= \Lambda \int \int \int_{V_0} \left(\rho_0 \eta \frac{\partial}{\partial X_A} (\delta Y_A) + \rho_0 \frac{\partial \eta}{\partial \tau} \Big|_{\tau=0} \right) dV_0 \\
 &= \Lambda \int \int \int_{V_0} \left(- \frac{\partial}{\partial X_A} (\rho_0 \eta) \delta Y_A + \rho_0 \delta y \right) dV_0 + \Lambda \int \int_S \delta Y_A (\rho_0 \eta) dS \\
 &\quad + \text{expressions on boundary } \partial v_0.
 \end{aligned} \tag{17}$$

Thus, following the Gibbs principle defining the minimum energy condition in the case of constant entropy, we can write:

$$\delta \left[\int \int \int_{V_0} \rho_0 \varepsilon \, dV_0 + \Lambda \int \int \int_{V_0} \rho_0 \eta \, dV_0 \right] = 0. \tag{18}$$

Considering the definition of the pseudomomentum tensor (chemical potential) in form of:

$$P_{AB} = \rho_0 (\varepsilon + \eta \Lambda) \delta_{AB} - F_{iB} T_{iA} \tag{19}$$

we obtain:

- on the side of the solid body:

$$\begin{aligned}
 &\int \int \int_{V_0} \left\{ \rho_0 \delta \eta \left(\frac{\partial \varepsilon}{\partial \eta} + \Lambda \right) - \delta y_i [T_{iA} \Lambda] + \frac{\partial}{\partial X_A} P_{AB} \delta Y_C \right\} dV_0 \\
 &+ \int \int_S \left[\delta y_i T_{iA} N_A + \delta Y_B P_{BA} N_A \right] dS + \int_{\partial v_0} (\delta y_i T_{iA} + \delta Y_B) P_{BA} N_A d(\partial V_0).
 \end{aligned} \tag{20}$$

- on the side of the fluid:

$$- \int \int \int_V \left\{ \rho_0 \frac{\partial \varepsilon}{\partial J} \frac{\partial J}{\partial y_i} \delta \left(\frac{\partial y_i}{\partial Y_B} \right) + \rho_0 \frac{\partial \varepsilon}{\partial \eta} \delta \eta + \rho_0 \varepsilon \frac{\partial}{\partial X_A} (\delta Y_A) \right\} dV_0. \tag{21}$$

For the whole region (liquid + solid) the following balance equations are obtained:

- momentum balance:

$$\delta y : \text{DIV } \vec{T} + \rho_0 \vec{b} = 0 \quad \text{on } V_S \cup V_1 \tag{22}$$

- pseudomomentum balance:

$$\delta Y : \quad \text{DIV } \vec{P} = 0 \quad \text{on } V_S \cup V_1 \tag{23}$$

together with two jump conditions on the interphase surface S :

- jump condition for the momentum flux

$$\delta y^i : \quad T_{iA} N_A|_{\text{solid}} + T_{iA} N_A|_{\text{liquid}} = 0$$

or:

$$\llbracket \vec{T} \rrbracket_i \vec{N} = 0, \quad (24)$$

- jump condition for the chemical potential tensor

$$\delta T_B : \quad P_{AB} N_A|_{\text{solid}} + P_{AB} N_A|_{\text{liquid}} = 0$$

or considering, that $N_A|_{\text{solid}} = -N_A|_{\text{liquid}}$

$$\llbracket \vec{P} \rrbracket_i \vec{N} = 0. \quad (25)$$

Following Gibbs [14], equilibrium of the chemical potential can be expressed as the chemical-potential tensor equilibrium in the normal direction \vec{N} to the rigid body.

$$[\rho_0(\varepsilon - \theta\eta) - N_A F_{iA} T_{iB} N_B]_{\text{solid}} - \left[\rho_0 \left(\varepsilon - \theta\eta + \frac{p}{\rho} \right) \right]_{\text{liquid}} = 0. \quad (26)$$

In the case of coherent phase transitions, the two remaining tangent components are not significant. The $F_{iA} T_{iB}$ term in Eq. (26) is the one that was not included in the original Gibbs' equation ([14], Eq. (369)). This element allows for better understanding of the role of the chemical potential.

5. Thermokinetic variational principle of Natanson (1899)

Natanson [15] has extended Gibbs' variational principle to cover the dynamic case by kinetic energy and mechanical forces inclusion. In this way, an extension of the classical definition of the chemical potential with the energy T and mass forces potential Ω was included.

$$\mu = \varepsilon - \theta\eta + pv - T + \Omega. \quad (27)$$

Moreover, unlike the Lagrangian Gibbs' approach, Natanson's approach is a purely Eulerian one, differing in the definition of interphase surface virtual motion.

Let us now consider the two volumes of the same fluid, divided by an interphase surface s , assuming that the fluid on both sides is in different phases (Fig. 2).

The thermokinetic Natanson principle can be written as:

$$\delta T + \delta F + \delta W + \delta' Q = 0. \quad (28)$$

Considering that the total kinetic energy is the sum of kinetic energy of all phases (neglecting kinetic energy of the interphase surface, as in this approach the interphase surface is a 'simple' dividing surface) and assuming that there is no slip between the phases (velocity of the ideal fluid transforming into the other phase is sufficiently similar to

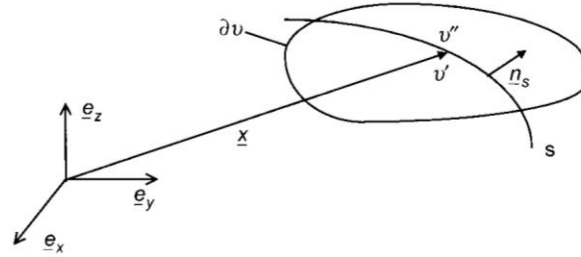


Fig. 2. Interphase geometry in the Eulerian representation.

potential flow velocity), we obtain that the variation of kinetic energy arising from 'natural inflows' into the volume v' bounded within the surface $\partial v'$ and containing the phase-dividing surface oriented outwards is equal to ([15], Eq. (5)):

$$\begin{aligned} \delta T' = \frac{d'}{dt} \left\{ - \iiint_{v'} \left[\varphi' \delta \rho' + \varphi' \nabla \rho' \cdot \delta \vec{x}' \right] dv' + \iint_{\partial v'} \rho' \varphi' \vec{n}' \cdot \delta \vec{x}' ds + \iint_s \rho' \varphi' \vec{n}'_s \cdot \delta \vec{x}' ds \right\} \\ - \left\{ \iiint_{v'} \left[\vartheta' \delta \rho' + \vartheta' \nabla \rho' \cdot \delta \vec{x}' \right] dv' + \iint_{\partial v'} \rho' \vartheta' \vec{n}' \cdot \delta \vec{x}' ds + \iint_s \rho' \vartheta' \vec{n}'_s \cdot \delta \vec{x}' ds \right\}, \end{aligned} \quad (29)$$

where ϑ is the Appel acceleration potential and φ is the velocity potential. An analogous expression is obtained for the variation of kinetic energy in the volume $\partial v''$ with reversed orientation of the normal vector, that is: $\vec{n}''_s = -\vec{n}'_s$.

Because the kinetic energy balanced within the volume cannot change, displacement through the interphase surface will transport the energy from the first system to the particles of the second one. Alternatively, we can say that the system v' will give back the following amount of energy as the result of infinitesimal change $D\vec{x}'$:

$$\begin{aligned} \frac{d'}{dt} \left\{ \iint_{\partial v'} \rho' \varphi' (\vec{n}' \cdot D\vec{x}') ds + \iint_s \rho' \varphi' (\vec{n}'_s \cdot D\vec{x}') ds \right\} + \iint_{\partial v'} \rho' \vartheta' (\vec{n}' \cdot D\vec{x}') ds \\ + \iint_s \rho' \vartheta' (\vec{n}'_s \cdot D\vec{x}') ds. \end{aligned} \quad (30)$$

A similar expression is valid for the system v'' .

Assuming that the phase transition of interest is isothermal, the variation of the free energy in the system v' can be described as:

$$\delta F' = \iint_{v'} \rho' \frac{\partial \psi'}{\partial \rho'} \delta \rho' dv' + \iint_{\partial v'} \rho' \psi' (\vec{n}' \cdot D\vec{x}') ds + \iint_s \rho' \psi' (\vec{n}'_s \cdot D\vec{x}') ds. \quad (31)$$

An analogous expression is obtained for the system v'' .

Due to the fact that the investigated system is forced by potential forces:

$$\vec{b}' = -\text{grad } \Omega' \quad \text{and} \quad \vec{b}'' = -\text{grad } \Omega''$$

and the boundary loads

$$\vec{f}' = -p'_v \vec{n}' \quad \text{and} \quad \vec{f}'' = -p''_v \vec{n}'',$$

the variation of the work done by these forces on virtual displacements $\delta \vec{x}'$ and $D\vec{x}'$ in system v' as well as $\delta \vec{x}''$ and $D\vec{x}''$ in system v'' can be written as:

$$\begin{aligned} \delta W' = & \int \int_{\partial v'} [\vec{f}'(\delta \vec{x}' + D\vec{x}') + \rho' \Omega' \vec{n}'(\delta \vec{x}' + D\vec{x}')] ds + \int \int_s \rho' \Omega' \vec{n}'_s(\delta \vec{x}' + D\vec{x}') \\ & ds + \int \int \int_{v'} \Omega'(\delta \rho' + \text{grad } \rho' \cdot \delta \vec{x}') dv' \end{aligned} \quad (32)$$

and analogously for the volume v'' .

The second principle of thermodynamics results in a non-negative increment of the uncompensated heat $\delta'Q$. As we consider only two fluids undergoing a reversible phase transition (without slip), we can take:

$$\delta'Q = 0. \quad (33)$$

The above leads to the variational formulation of the phase transition equilibrium. Taking into account:

- no-slip condition on the interphase surface,

$$\int \int_s \vec{D}(D\vec{x}' - D\vec{x}'') ds = 0, \quad (34)$$

- neighborhood-preserving condition for interphase surface particles

$$\int \int_s \vec{A}(\delta \vec{x}' - \delta \vec{x}'') ds = 0, \quad (35)$$

- mass-preserving condition:

$$\begin{aligned} & \int \int \int_{v'} [B' \delta \rho' - \text{grad}(\delta \vec{B}' \cdot \delta \vec{x}')] dv' - \int \int_{\partial v'} \rho' B' \vec{n}' \cdot \delta \vec{x}' ds \\ & - \int \int_s \rho' B' \vec{n}'_s \cdot \delta \vec{x}' ds = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} & \int \int_{\partial v'} \rho' C \vec{n}' \cdot D\vec{x}' ds' + \int \int_{\partial v''} \rho'' C \vec{n}'' \cdot D\vec{x}'' ds'' + \int \int_s \rho' C \vec{n}'_s \cdot D\vec{x}' ds \\ & + \int \int_s \rho'' C \vec{n}''_s \cdot D\vec{x}'' ds = 0, \end{aligned} \quad (37)$$

where $\vec{A}, B' (t, \vec{x}), \vec{D}$ are Lagrange's multipliers and $C = C(t)$ is any function of time, finally the Natanson principle is obtained as:

$$\int_{t_0}^{t_1} [I_{v'} + I_{v''} + \partial I_{v'} + \partial I_{v''} + S] dt = 0, \quad (38)$$

where

$$\begin{aligned}
I_{v'} : & - \int \int \int_{v'} \left[\vartheta' \delta \varphi' - \vartheta' \nabla \rho' \cdot \delta \vec{x}' + \rho' \frac{\partial \psi'}{\partial \rho'} \delta \rho' + \Omega' \delta \rho' \right] dv' \\
& + \int \int \int_{v'} \left[-\Omega' \nabla \rho' \cdot \delta \vec{x}' - B' \delta \rho' + \nabla(\rho' B') \cdot \delta \vec{x}' \right] dv' \\
I_{v''} : & - \int \int \int_{v''} \left[\vartheta'' \delta \varphi'' - \vartheta'' \nabla \rho'' \cdot \delta \vec{x}'' + \rho'' \frac{\partial \psi''}{\partial \rho''} \delta \rho'' + \Omega'' \delta \rho'' \right] dv'' \\
& + \int \int \int_{v''} \left[\Omega'' \nabla \rho'' \cdot \delta \vec{x}'' - B'' \delta \rho'' + \nabla(\rho'' B'') \cdot \delta \vec{x}'' \right] dv'' \\
\partial I_{v'} : & \int \int_{\partial v'} \left[\rho' \vartheta' \vec{n}' \cdot (\delta \vec{x}' + D \vec{x}') + \rho' \psi' \vec{n}' \cdot D \vec{x}' + \vec{f}' \cdot (\delta \vec{x}' + D \vec{x}') \right] ds' \\
& + \int \int_{\partial v'} \left[\rho' \psi' \vec{n}' \cdot (\delta \vec{x}' + D \vec{x}') - \rho' B' \vec{n}' \cdot \delta \vec{x}' + \rho' C \vec{n}' \cdot D \vec{x}' \right] ds' \\
\partial I_{v''} : & \int \int_{\partial v''} \left[\rho'' \vartheta'' \vec{n}'' \cdot (\delta \vec{x}'' + D \vec{x}'') + \rho'' \psi'' \vec{n}'' \cdot D \vec{x}'' + \vec{f}'' \cdot (\delta \vec{x}'' + D \vec{x}'') \right] ds'' \\
& + \int \int_{\partial v''} \left[\rho'' \psi'' \vec{n}'' \cdot (\delta \vec{x}'' + D \vec{x}'') - \rho'' B'' \vec{n}'' \cdot \delta \vec{x}'' + \rho'' C \vec{n}'' \cdot D \vec{x}'' \right] ds'' \\
S : & \int \int_s \left[\rho' \vartheta' \vec{n}'_s \cdot (\delta \vec{x}' + D \vec{x}') + \rho'' \vartheta'' \vec{n}''_s \cdot (\delta \vec{x}'' + D \vec{x}'') + \rho'' \psi'' \vec{n}''_s \cdot D \vec{x}'' \right] ds \\
& + \int \int_s \left[\rho' \psi' \vec{n}'_s \cdot D \vec{x}' + \rho' \Omega' (\vec{n}'_s \cdot \delta \vec{x}' + \vec{n}'_s \cdot D \vec{x}') + \rho'' \Omega'' (\vec{n}''_s \cdot \delta \vec{x}'' + \vec{n}''_s \cdot D \vec{x}'') \right] ds \\
& + \int \int_s \left[\vec{A} \cdot (\delta \vec{x}' + \delta \vec{x}'') - \rho' B' \vec{n}'_s \cdot \delta \vec{x}' - \rho'' B'' \vec{n}''_s \cdot \delta \vec{x}'' \right] ds \\
& + \int \int_s \left[\vec{D} \cdot (D \vec{x}' + D \vec{x}'') + \rho' C \vec{n}'_s \cdot D \vec{x}' + \rho'' C \vec{n}''_s \cdot D \vec{x}'' \right] ds,
\end{aligned}$$

This will lead us to the following equations on the surface s :

$$\delta \vec{x}' : \quad \rho' (\vartheta' + \Omega' - B') \vec{n}'_s + \vec{A} = 0, \quad (39)$$

$$\delta \vec{x}'' : \quad \rho'' (\vartheta'' + \Omega'' - B'') \vec{n}''_s - \vec{A} = 0, \quad (40)$$

$$D \vec{x}' : \quad \rho' (\vartheta' + \psi' + \Omega' - C) \vec{n}'_s + \vec{D} = 0, \quad (41)$$

$$D \vec{x}'' : \quad \rho'' (\vartheta'' + \psi'' + \Omega'' - C) \vec{n}''_s + \vec{D} = 0. \quad (42)$$

Eqs. (39) and (40) lead us, as expected, to the second Gibbs' condition:

$$\delta \vec{x} : \quad \vec{n}'_s (p' - p'') = 0 \quad \text{on the surface } s, \quad (43)$$

that is $p' = p''$.

Because the extended third Gibbs' condition is in the form of:

$$D\vec{x}\cdot\vec{n}_s : \quad \zeta' + \vartheta' + \Omega' = \zeta'' + \vartheta'' + \Omega'', \quad (44)$$

where $\zeta' = \psi' + p'v'$ and $\zeta'' = \psi'' + p''v''$ are free enthalpy, Eqs. (41) and (42) can be written as the jump condition:

$$[\![\psi + pv]\!]'' + [\![\vartheta]\!]'' + [\![\Omega]\!]'' = 0. \quad (45)$$

The presence of jump $[\![\vartheta]\!]$ allows for description of the phase transition in the flow, whereas $[\![\Omega]\!]$ takes into account the presence of mass forces.

The third Gibbs' condition has not so far received a simple interpretation, even in the case of homogenous phase transition. The literature has been dominated by the interpretation based upon Natanson's reasoning, which reads the third Gibbs' condition as a zero-entropy production requirement (that is the condition for phenomena reversibility) simplified after the heat equilibrium condition was incorporated into the expression for entropy production.

6. The Eckart variational principle

6.1. Variational statement of pseudomomentum conservation

The Lagrangian description of the flow coupled with Lagrangian variation of particle displacements provides the most natural formalism. It is a natural extension of the most powerful methods introduced in analytical mechanics by Lagrange, Hamilton and Jacobi. Most of the older work on this direction (R. Gwyther [52], A. Clebsch, L. Lichtenstein, H. Bateman, W. Weber) touch on the problem of pseudomomentum conservation, but the most excellent account of this approach is to be found in the paper by Eckart [16].

The paper under reconstruction starts from a quite general Lagrangian variational problem of finding $x_i (i = 1, 2, \dots, m)$ functions of the independent variables $X_A (A = 1, 2, \dots, N)$. The fundamental object is the Lagrangian density function L^m (J/m^3), i.e. by definition, the kinetic energy minus the potential energy density:

$$L^m = L^m(X_A; x_i, x_{i,A}), \quad A = \tau, X, Y, Z, \quad (46)$$

where a convenient notation $x_{i,A} = \partial x_i / \partial X_A$ is used. The upper index m (as: material) informs us that density is referred to the undeformed (initial) volume. Since all the forms of energy depend only on first derivatives (viscosity and thermal conductivity are omitted) $x_{i,A}$ we have to deal with the so-called 'first-grade continuum', i.e. the simple continuum. If we consider the action integral I ([16], Eq. (2.3))

$$I = \int \int \dots \int L^m dX_1 dX_2 \dots dX_N \quad (47)$$

over the N -dimensional volume (in time-space $N = 1, X_1 = \tau, N = 2, 3, 4, X_2, X_3, X_4 = X, Y, Z$). The variational Euler-Lagrange equations resulting from $\delta I = 0$

in case of fixed boundary problems, are [1]:

$$\frac{\partial}{\partial X_A} \left(\frac{\partial L^m}{\partial x_{i,A}} \right) - \frac{\partial L^m}{\partial x_i} = 0, \quad i = 1, 2, 3, \dots, m. \quad (48)$$

Eqs. (48) are called field equations or ‘equations of motion’ for fields x_i associated with the action integral (47). Lagrangian (46) is usually connected with a few conservation laws that, according to the Noether theorem, are based on certain groups of transformations of independent (X_A) and dependent variables (x_i). Eckart proposed a simple method to obtain the conservation laws. Differentiating L^m with respect to X_A he obtained the pseudoenergy-momentum conservation in the form ([16], Eq. (2.5)):

$$\frac{\partial}{\partial X_B} L_{AB} = - \left(\frac{\partial L^m}{\partial X_A} \right)_{\text{exp}}, \quad (49)$$

where the pseudoenergy–momentum complex L_{AB} (in the case $N = 4$; X, Y, Z , it has a 4×4 array representation) is defined to be:

$$L_{AB} = x_{i,A} \frac{\partial L^m}{\partial x_{i,B}} - L^m \delta_{AB}. \quad (50)$$

Note that we have to distinguish the explicit partial derivative L^m with respect to X_A , when its other arguments x_i , $x_{i,A}$ and remaining $X_B, A \neq B$, are kept constant. Nowadays it is denoted by [48,54]:

$$\left(\frac{\partial L^m}{\partial X_A} \right)_{\text{exp}} = \frac{\partial L^m(x_i, x_{i,A}; X_A, X_B)}{\partial X_A} \Big|_{x_i, x_{i,A}, X_B \neq A}. \quad (51)$$

So far Eqs. (49)–(51) are purely mathematical relations. Eq. (49) would become conservation laws if its right-hand side were to vanish, i.e. if the Lagrangian did not depend explicitly on independent variables. This happens if the fluid medium is in the state of pure homogeneity with respect to space (X, Y, Z) and time (τ). For defects like vortices, shock waves, surface of discontinuity, surface of phase transitions, etc., which can be treated as inhomogeneity in the fluid medium, the term $(\partial L^m / \partial X_A)_{\text{exp}}$ has an interpretation of material of configurational force (if $A = X, Y, Z$) or material energy source (if $A = \tau$).

6.2. The incompressible fluid

In this case the Lagrangian consists solely of the kinetic specific energy T , and there is no internal energy. Trajectories of fluid particles are described by specifying three functions of position $\vec{x} = x_i \vec{e}_i$, $i = x, y, z$:

$$x_i = x_i(X, Y, Z, \tau) = x_i(X_A, \tau). \quad (52)$$

The derivatives in Eq. (46) now have their own names, respectively: the velocity $\vec{v} = v_i \vec{e}_i$:

$$v_i = \frac{\partial}{\partial \tau} x_i = x_{i,\tau}(X_A(x_i, t), \tau) \quad (53)$$

and the gradient of deformation $\vec{F} = F_{iA} \vec{e}_i \otimes \vec{E}_A = \text{GRAD} \vec{x}$:

$$F_{iA} = x_{i,A} = \frac{\partial x_i}{\partial X_A}. \quad (54)$$

In this case the coordinates X, Y, Z also play the role of Lagrangian position vector $\vec{X} = X_A \vec{E}_A$. Inverting the observer, one obtains the inverted deformation gradient $\vec{f} = f_{Ai} \vec{E}_A \otimes \vec{e}_i = \text{grad} \vec{X}$ written as:

$$f_{Ai} = X_{A,i} = \frac{\partial X_A}{\partial x_i}, \quad F_{iA} f_{Aj} = \delta_{ij}, \quad f_{Ai} F_{iB} = \delta_{AB}. \quad (55)$$

Let us denote by J and j the determinant of \vec{F} and \vec{f} , respectively:

$$J = \det(x_{i,A}), \quad j = \det(X_{A,i}), \quad J = j^{-1}, \quad (56)$$

then, the following Euler–Piola–Jacobi identity takes place ([16], Eq. (2.2)):

$$(j x_{i,A})_{,i} = \text{div}(j \vec{F}) = 0, \quad (J X_{A,i})_{,A} = \text{DIV}(J \vec{f}) = 0, \quad \frac{\partial J}{\partial x_{i,A}} = J X_{A,i}, \quad (57)$$

$$dv = J dV.$$

The connections between the Lagrangian and Eulerian changes will be explained as follows:

$$\left. \frac{\partial X_A}{\partial \tau} \right|_{\vec{x} \text{ fixed}} = \left. \frac{\partial X_A}{\partial t} \right|_{\vec{x} \text{ fixed}} + \left. \frac{\partial x_i}{\partial \tau} \right|_{\vec{x} \text{ fixed}} \frac{\partial X_A}{\partial x_i} = V_A + v_i X_{A,i} \equiv 0 \quad (58)$$

$$\left. \frac{\partial x_i}{\partial t} \right|_{\vec{x} \text{ fixed}} = \left. \frac{\partial x_i}{\partial \tau} \right|_{\vec{x} \text{ fixed}} + \left. \frac{\partial X_A}{\partial t} \right|_{\vec{x} \text{ fixed}} \frac{\partial x_i}{\partial X_A} = v_i + V_A x_{i,A} \equiv 0. \quad (59)$$

Here, $\vec{v} = v_i \vec{e}_i$ and $\vec{V} = V_A \vec{E}_A$ denote velocity and the rate of flow of material, or briefly, the material velocity.¹ The Lagrangian L^m contains, apart from kinetic energy T and gravity potential $\Omega(x_i)$, also an additional constraint related with incompressibility of fluid:

$$L^m = \rho_0 [T(x_{i,\tau}) - \Omega(x_i)] + P[J(x_{i,A}) - \rho_0/\rho], \quad (60)$$

where the former Lagrange multiplier $P(X_A)$ turns out to be the pressure.

The Euler–Lagrange Eqs. (60) according to Eq. (48) are [16, Eq. (3.3)]:

$$\rho_0 x_{i,\tau\tau} + J P_{,A} X_{A,i} + \rho_0 f_i = 0, \quad J \rho = \rho_0, \quad (61)$$

where the physical force density f_i is defined as $f_i = \partial \Omega / \partial x_i|_{\text{exp}}$.

The pseudoenergy-momentum tensor Eq. (50) now is simply determined as ([16], Eq. (3.8))

$$\text{Pseudoenergy} \quad L_{\tau\tau} = E = \rho_0(T + Q), \quad (62)$$

$$\text{Pseudoenergy flux} \quad L_{\tau A} = E_A = P X_{A,i} v_i = -P V_A, \quad (63)$$

$$\text{Pseudomomentum} \quad L_{A\tau} = P_A = \rho_0 v_i x_{i,A} = \rho_0 u_A, \quad (64)$$

$$\text{Pseudomomentum flux} \quad L_{AB} = P_{AB} = \rho_0(P/\rho - T)\delta_{AB}. \quad (65)$$

¹ For the definition see: L. Natanson (1986) On laws of irreversibility phenomena, Phil. Mag. 385–406, 41.

In Eq. (64), Eckart denotes the specific pseudomomentum vector (m/s per unit mass) by u_A . This definition is preferred in the literature of atmosphere and ocean physics [8,43–46, 55,56]. From the definition of Eq. (61), since the Lagrangian is explicitly independent of X_A and τ , it follows that the configurational force (51) both in space and time components are zero. Therefore, Eqs. (49) and (61)–(65) result in the conservation laws of pseudoenergy and pseudomomentum, respectively ([16], Eqs. (3.10) and (3.11))

$$\frac{\partial}{\partial \tau} E + \frac{\partial}{\partial X_A} (E_A) = \frac{\partial}{\partial \tau} \rho_0 (T + \Omega) + \frac{\partial}{\partial X_A} (-P V_A) = 0, \quad (66)$$

$$\frac{\partial}{\partial \tau} P_A + \frac{\partial}{\partial X_B} (P_{AB}) = \frac{\partial}{\partial \tau} \rho_0 u_A + \frac{\partial}{\partial X_A} \left(\frac{P}{\rho} + \Omega - T \right) = 0. \quad (67)$$

Using Eqs. (57) and (59), the balance of pseudoenergy Eq. (66) can be transformed to ([16], Eq. (3.14))

$$\frac{\partial}{\partial \tau} \rho_0 (T + P/\rho) = \epsilon^{\alpha\beta\gamma\delta} P_{,\alpha} x_{1,\beta} x_{2,\gamma} x_{3,\delta} \quad \alpha, \beta, \gamma, \delta = \tau, X, Y, Z. \quad (68)$$

This is a general form of the Bernoulli theorem. In the case of steady motion P is independent of time and the right side Eq. (68) vanishes, and Eq. (68) integrates at once to the usual form well known from textbooks. After the differentiation in Eq. (67) has been performed, the pseudomomentum conservation reduces to:

$$\rho_0 x_{i,\tau\tau} x_{i,A} + P_{,A} = 0, \quad (69)$$

which is the form proposed by Lamb,² who derived Eq. (69) using the Euler–Lagrange equations of motion multiplied by \vec{f} . Such an approach is correct only if the Lagrangian contains one dependent function, i.e. $x_i(X_A, \tau)$.

The balance of pseudomomentum can be implemented for evaluation of the vortex conservation and vortex coordinates. Let us recall that for a closed curve Γ_0 in the initial position \vec{X} the circulation is defined to be:

$$C(\tau) = \oint_{\Gamma_0} \vec{u} \cdot d\vec{X} = \oint_{\Gamma_0} u_A dX_A \quad (70)$$

while from Eq. (67) and the constancy of curve Γ following is obtained ([16], Eq. (4.2))

$$\frac{\partial}{\partial \tau} C(\tau) = \oint_{\Gamma_0} d \left(\frac{P}{\rho} + \Omega - T \right) = 0. \quad (71)$$

It is known as the Kelvin–Helmholtz circulation theorem. If $\Gamma(\tau)$ is the instantaneous position and shape of a moving closed curve that is always composed of the same particles (i.e. is ‘fixed in the fluid’) then the circulation is described by:

$$C(\tau) = \oint_{\Gamma(\tau)} \vec{v} \cdot d\vec{x} = \oint_{\Gamma(\tau)} v_i dx_i. \quad (72)$$

² H. Lamb (1945) *Hydrodynamics*, Dover, New York, p. 13.

From Eq. (71) and from Kelvin's vorticity theorem:

$$\oint_{\Gamma} \vec{u} \cdot d\vec{X} = \iint_S \text{ROT} \vec{u} \cdot d\vec{S} = \iint_S \vec{W} \cdot d\vec{S} \quad (73)$$

it follows that the material vorticity vector

$$\vec{W} = \text{ROT} \vec{u} = (u_{Z,Y} - u_{Y,Z}) \vec{E}_X + (u_{X,Z} - u_{Z,X}) \vec{E}_Y + (u_{Y,X} - u_{X,Y}) \vec{E}_Z \quad (74)$$

is conserved in time

$$\frac{\partial}{\partial \tau} \vec{W} = 0, \quad (75)$$

which means that \vec{W} is a function of X_A only, independent of τ . Then, according to Clebsch's assertions it is always possible to find three scalars A, B, ϕ such that

$$\vec{u} = A \text{GRAD } B - \text{GRAD } \phi, \quad (76)$$

which leads to an expression of the material vorticity

$$\vec{W} = (\text{GRAD } A) \times (\text{GRAD } B). \quad (77)$$

When $A = X, B = Y$ such a coordinate system is called the vortex coordinates.

6.3. The compressible perfect fluid

In the case of isenthalpic motion of a compressible fluid, an internal specific energy $\varepsilon = \varepsilon(\nu, \eta)$ should be added to the Lagrangian. According with Gibbs' notation [14]— ν is specific volume, $\nu = \rho^{-1}$ and η specific entropy. For this hyperelastic medium the constitutive equations for the thermodynamic pressure p and the thermodynamic temperature θ are given as:

$$p = - \left. \frac{\partial \varepsilon}{\partial \nu} \right|_{\eta} = \rho^2 \frac{\partial \varepsilon}{\partial \rho}, \quad \theta = \left. \frac{\partial \varepsilon}{\partial \eta} \right|_{\nu}. \quad (78)$$

The dependence of density ρ on the deformation gradient is only via the Jacobian $J = \det(x_{i,A})$. The requirement that the motion be isenthalpic is formulated by a new subordinate condition ([16], Eq. (3.3)):

$$\frac{\partial}{\partial \tau} \eta = 0, \quad (79)$$

which means that η is a function only of X_A , $A = X, Y, Z$. It may be introduced into L^m at the beginning, so that entropy η is not a dependent variable but a given function of X_A independent of τ .

The Euler–Lagrange equations for a problem given by the Lagrangian function

$$L^m = \rho_0 (T - \varepsilon - \Omega - \theta \eta) \quad (80)$$

are given as ([16], Eq. (3.5))

$$\rho_0 x_{i,\tau\tau} + J p_{,A} X_{A,i} + \rho_0 f_i = 0, \quad \theta = \frac{\partial \varepsilon}{\partial \eta}, \quad (81)$$

where p is defined via Eq. (78). According to definition (50) the pseudoenergy and pseudomomentum conservation are described by ([16], Eq. (5.5)–(5.8))

$$\text{Pseudoenergy} \quad L_{\tau\tau} = E = \rho_0(T + \Omega + \varepsilon + \theta\eta), \quad (82)$$

$$\text{Pseudoenergy flux,} \quad L_{\tau A} = E_A = P X_{A,i} v_i = -p V_A, \quad (83)$$

$$\text{Pseudomomentum} \quad L_{A\tau} = P_A = \rho_0 v_i x_{i,A} = \rho_0 u_A, \quad (84)$$

$$\text{Pseudomomentum flux} \quad L_{AB} = P_{AB} = \rho_0(\varepsilon + P/\rho - \theta\eta - T)\delta_{AB}. \quad (85)$$

This leads to the following pseudoenergy and pseudomomentum conservation:

$$\frac{\partial}{\partial \tau} \rho_0(T + \Omega + \varepsilon + \theta\eta) + \frac{\partial}{\partial X_A} (-p V_A) = 0, \quad (86)$$

$$\frac{\partial}{\partial \tau} \rho_0 u_A + \frac{\partial}{\partial X_A} \rho_0 \left(\varepsilon + \frac{p}{\rho} + \Omega - \theta\eta - T \right) = 0. \quad (87)$$

Using the material flow u_A , and the material vorticity vector W_A , Eckart was able to prove the Helmholtz–Bjerknes theorem that asserts that the motion will be irrotational if the pressure can be expressed as a function of specific volume only—i.e. whenever the medium possesses the barotropic constitutive equation.

Eckart has also introduced a somewhat remarkable definition of a thermal potential κ ([16], Eq. (6.15)):

$$\frac{\partial}{\partial \tau} \kappa = \theta, \quad (88)$$

which leads to a novel definition of the thermodynamic circulation around the curve Γ as:

$$C_{T\kappa} = - \oint_{\Gamma} \kappa d\eta \quad (89)$$

and the kinematical circulation C_{ki} as in Eq. (70). Then, interpreting pseudomomentum conservation (87) Eckart asserts that the total circulation is conserved:

$$\frac{\partial}{\partial \tau} (C_{T\kappa} + C_{ki}) = 0. \quad (90)$$

The total circulation, defined in this way, remains constant for every closed curve Γ fixed in the particles of fluid.

7. Eulerian representation of pseudomomentum

It is well known that the balance of momentum and energy are usually postulated within the framework of Newtonian mechanics and Joulean thermodynamics, respectively.

Most frequently the responsible flux of momentum:

$$\pi_{ij} = \pi_i v_j + t_{ij} - \tau_{ij} + \dots \quad (91)$$

and the flux of energy

$$e_i = e v_i + (t_{ij} - \tau_{ij} + \dots) v_j - q_i^\eta \theta + \dots \quad (92)$$

are postulated, especially in the case when it is phenomenogically difficult to describe an amount of momentum and energy carried by additional fields like: radiation, progress of phase transition, mass fraction, etc. Quite similar uncertainty occurs when the medium is described via higher-order derivatives [4,5]. Therefore, looking for more precisely defined fluxes of momentum and energy, it is more comfortable to start from the variational principle based on the spatial (Eulerian) Lagrangian, say L^s [11]. Then, four equations of balance of momentum and energy can be obtained independently from the Euler–Lagrange ‘equation of motion’. Both momentum and energy balances have their representations in the Lagrangian description. For instance, the flux of momentum for elastic fluid is called the Cauchy flux and its Lagrangian representation—the 1st Piola–Kirchhoff flux. Quite a similar situation concerns the pseudomomentum flux (sometimes called the Eshelby flux)—its Eulerian representation should be defined and useful in describing a few subjects stated in the Eulerian description (for instance; radiation flux pressure [39], radiation flux stress [8]).

To understand a relation between momentum and pseudomomentum in both Lagrangian and Eulerian representations, let us consider a very simple case, purely mechanical when the positions $\tilde{X} = X_A \tilde{E}_A$ and $\tilde{x} = x_i \tilde{e}_i$ are solely dependent and independent functions and exchange their role in passing from the Lagrange to the Euler description and vice versa.

Taking:

$$L^m = L^m(\tau, X_A; x_i, x_{i,A}, x_{i,\tau}, x_i) \quad (93)$$

$$L^s = L^s(t, x_i; X_{A,i}, X_{A,t}, X_A) \quad (94)$$

and using both the Euler–Lagrange equation and pseudoenergy-momentum conservation we obtain the following set of equations [48]:

Eulerian description:

$$\text{Momentum conservation:} \quad \partial_t \pi_i + \pi_{ij,j} = \left(\frac{\partial L^s}{\partial x_i} \right) \Big|_{\text{exp}}, \quad (95)$$

$$\text{Pseudomomentum (E–L eq.)} \quad \partial_t p_A + p_{Aj,j} = - \left(\frac{\partial L^s}{\partial X_A} \right) \Big|_{\text{exp}}, \quad (96)$$

Lagrangian description:

$$\text{Momentum (E–L eq.):} \quad \partial_\tau \Pi_i + \Pi_{iA,A} = \left(\frac{\partial L^m}{\partial x_i} \right) \Big|_{\text{exp}}, \quad (97)$$

$$\text{Pseudomomentum conservation} \quad \partial_\tau P_A + P_{AB,B} = - \left(\frac{\partial L^m}{\partial X_A} \right) \Big|_{\text{exp}}. \quad (98)$$

Above, a simple system of notation has been applied:

- letters π, Π (small, capital) always denote momentum (vector, tensor),
- letters p, P (small capital) always denote the pseudomomentum (vector, tensor),
- small letters (π, p) indicate on the Eulerian description,
- capital letters (Π, P) indicate on the Lagrangian description.

In the recent literature many different denotations of the above objects exist. The most frequently used ones are collected in Table 1.

Let us describe the above objects in terms of derivatives L^s and L^m , respectively. The physical momentum is defined as:

$$\pi_i = \partial L^s / \partial v_i = \rho v_i \quad (99)$$

$$\pi_{ij} = \pi_i v_j + t_{ij}, \quad t_{ij} = x_{i,A} (\partial L^s / \partial x_{j,A}) - L^s \delta_{ij}, \quad (100)$$

$$\Pi_i = \partial L^m / \partial v_i = \rho_0 v_i = J \pi_i, \quad (101)$$

$$\Pi_{iA} = \partial L^m / \partial x_{i,A} = J t_{ij} X_{A,j}. \quad (102)$$

The pseudomomentum is analogously defined as:

$$p_A = \partial L^s / \partial V_A = j P_A, \quad (103)$$

$$p_{Ai} = \partial L^s / \partial X_{A,i} = j P_{AB} X_{B,i}, \quad (104)$$

$$P_A = \partial L^m / \partial v_i x_{i,A}, \quad (105)$$

$$P_{AB} = x_{i,A} (\partial L^m / \partial x_{i,B}) - L^m \delta_{AB}. \quad (106)$$

Using the above definitions, identities (57) and $L^s = j L^m$ it is simple to show (with straightforward calculations) direct relations between Lagrangian and Eulerian

Table 1

Most commonly used denotations of pseudomomentum

Author	Pseudomomentum flux tensor		Pseudomomentum vector	
	Lagrangian	Eulerian	Lagrangian	Eulerian
Eckart [16]	$\alpha_{\alpha\alpha}$	—	$u_\alpha, u_\beta, u_\gamma^a$	—
Eshelby [25]	P_{ij}	Σ_{ij}	g_i	—
Rogula [48]	\bar{p}_{AB}	p_{Ai}	\bar{p}_A	p_A
Andrews and McIntyre [17]	B_{ij}	—	A_i^a	—
Gołębiewska–Hermann [54]	b_{ij}	B_{ik}^b	b_i	B_i
Grinfeld [26]	μ_{IJ}	χ_{ij}^b	—	—
Peierlis [53]	—	—	κ_i^c	—
Duan [57]	$B^{\mu\nu}$	b_{iv}	B_ν	b_ν
Maguin and Trimarco [23]	b_{kl}	B_{ki}	P_k	$j P_k$
Gurtin [24]	C_{AB}	—	—	—
Present chapter	P_{AB}	p_{Ai}	P_A	p_A

^a Per unit mass.

^b Symmetric—an analog of the second Piola–Kirchhoff flux. Other applications of the symmetric chemical potential tensor are to be found in papers by McLellan [58] and Stuke [27].

^c In the whole volume.

representations in the following form [54]:

$$\frac{\partial}{\partial t} \pi_i + \pi_{ij,j} = -j \left(\frac{\partial}{\partial \tau} \Pi_i + \Pi_{iA,A} \right), \quad (107)$$

$$\frac{\partial}{\partial \tau} P_A + P_{AB,B} = -J \left(\frac{\partial}{\partial t} p_A + p_{Ai,i} \right). \quad (108)$$

Quite similarly, the Eulerian representation of the material (configurational) force can be now defined using Eq. (96). In this description the Eulerian configurational force density per unit volume is defined as

$$f_A^{\text{inh}} \equiv \partial_i p_A + p_{Ai,i} = -(\partial L^s / \partial X_A) \big|_{\text{exp}}, \quad (109)$$

whereas the Lagrangian configurational force density (per unit of undeformed volume) is defined to be:

$$F_A^{\text{inh}} \equiv \partial_\tau P_A + P_{AB,B} = -(\partial L^m / \partial X_A) \big|_{\text{exp}}, \quad (110)$$

where, $f_A^{\text{inh}} = j F_A^{\text{inh}}$ owing to $L^s = j L^m$. This means that the above definitions take an exact form of the Newtonian (physical) force definitions, except that we put pseudomomentum vector and pseudomomentum flux in the places of momentum vector and momentum flux tensor, respectively [59]. The physical dimensions of momentum and pseudomomentum are the same. The fundamental difference is that physical (Newtonian) forces cause things to move and accelerate in physical space, while the configurational (material) forces will do the same with a material inhomogeneity in the material space [48].

8. Capillarity extended Gibbs principle

The next chapter of Gibbs' memoir [14] is devoted to some supplementation of the model presented in Section 4 concerning the treatment of heterogeneous masses in contact. Let us recall, in Section 4 masses are in contact through a 'naked' surface of discontinuity, such that each mass is unaffected by the vicinity of the others. However, introducing a geometrical surface called the dividing surface, Gibbs had to take into account an anisotropy that occurs in the vicinity of the surface of discontinuity. Therefore, this surface possesses some superficial densities of energy ε^s specific entropy η^s and surface specific 'volume' S^s , surface specific curvatures c^s , etc. ([14], pp. 232–232).

The superficial energy density represents some excess of energy that can be added to the Lagrangian (12):

$$L^{\text{surf}} = \gamma \varepsilon^s(\eta^s, S^s, c^s), \quad (111)$$

where γ represents pseudomass density measured on a unit area of the surface. Owing to the excess of energy (111) the jump of pseudomomentum now takes the well-known form of the Laplace equation ([14], Eq. (500))

$$\delta N : \quad \sigma(c_1 + c_2) = p' - p'', \quad (112)$$

where $\sigma = (\partial \varepsilon / \partial S^s) \big|_{\eta^s, c^s}$ is the surface tension. Buff has extended the calculation of the dividing surface variation δN on the curvature tensor and has obtained the

pseudomomentum jump (112) not only with a surface tension $\vec{\sigma} = \sigma(\vec{I} - \vec{N} \otimes \vec{N})$ but also with the surface moment $\vec{C} = C(\vec{I} - \vec{N} \otimes \vec{N})$ ([60], Eq. (22)); see also Gibbs ([14], pp. 232, 232):

$$\sigma(c_1 + c_2) - (c_1^2 + c_2^2) \frac{C}{\gamma} = p' - p''. \quad (113)$$

Recently, the papers [61,62] have shown the Laplace equation to be a normal component of surface pseudomomentum conservation:

$$\delta \text{iv}[\vec{\sigma} - \vec{b}\vec{C} + \vec{N} \otimes (\vec{I} - \vec{N} \otimes \vec{N}) \delta \text{iv}(\vec{C} - \vec{b}\vec{K})] = (\vec{P}' - \vec{P}'')\vec{N}, \quad (114)$$

where P'_{AB}, P''_{AB} are the bulk pseudomomentum tensor, $\vec{\sigma}, \vec{C}, \vec{K} = K(\vec{I} - \vec{N} \otimes \vec{N})$ are surface pseudomomentum fluxes, tension, moment and bimoment, respectively. The curvature tensor is defined as $\vec{b} = \gamma \text{rad } \vec{N} = \vec{N} \otimes \nabla_2$. Two-dimensional divergence is denoted $\delta \text{iv}(\cdot) = (\cdot) \cdot \nabla_2$. When \vec{C} and \vec{K} vanish the above equation coincides with one developed by Gurtin ([24], Eq. (4.2)). Our notation is based on the fundamental notions concerning physical surfaces stated by Skiba [63] and Weatherburn [64], and developed intensively in the papers by Tolman [65] and Buff [60]. The next recent generalizations of Eq. (112) are to be found in the papers by Scriven [66], Gurtin [24] and Povstenko [67]. Some important applications can be found in papers by Konorski [68] and Puzyrewski et al. [69], as well as Geurst [70] and Truskinovsky [22]. In order to examine further implications of the extended Gibbs principle it should be noted that keeping carefully to Gibbs' line of reasoning one can obtain the Laplace equation as a special simplification of the pseudomomentum jump across the dividing surface. Let us note that there are many papers (for instance see [71]) where the Laplace equation is treated as a jump of physical momentum.

Additionally, let us note that the configurational force, in comparison with Eq. (26), is differently stated for both cases ('naked' and 'superficial' surface). If we denote by F^{inh} the normal component of the configurational force that is the main 'driving force' (not the Newtonian one!) as desertion from equilibrium given by the third Gibbs' condition:

$$F^{\text{inh}} = \begin{cases} \mu' - \mu'' \\ N_A(P'_{AB} - P''_{AB})N_B + \vec{N} \cdot (\vec{\sigma} \cdot \nabla_2) \end{cases} \quad (115)$$

for the case of 'naked' and 'superficial' surface, respectively. In the last formula we have omitted \vec{C} and \vec{K} tensors and a general form of $\vec{\sigma}$ can contain also traversal components as Gurtin has postulated ([24], Eq. (33)).

9. The acoustic pseudomomentum

Starting from research by Rayleigh [36] and Schrödinger [37] it is a well-known fact that using Lagrangian and Eulerian descriptions for an acoustic-wave statement leads to numerous discrepancies in results, such as in the case of, for instance, Rayleigh and Langevin radiation pressure [38–40]. Considerable confusion has arisen as to the precise meaning of the term 'radiation pressure' and its relevance to measurements. Even in the case when the basic fluid that is transmitting a wave is motionless, there are two

different—Lagrangian and Eulerian—definitions of acoustic displacements:

$$\vec{\xi} = \xi_0 \sin(\Omega\tau - \vec{K} \cdot \vec{X}), \quad \vec{\xi} = \xi_0 \sin(\omega t - \vec{k} \cdot \vec{x}), \quad (116)$$

which leads to different connections between wave (pseudo) momentum and wave energy. Let us recall that the most basic relations of this kind has been worked out by Rayleigh ([36], Eq. (23)):

$$\text{wave momentum} = \frac{3}{4} \frac{\text{total wave energy}}{c} \quad (117)$$

where c is the speed of wave propagation.

The question of how to formulate more general equations of acoustics for an arbitrary moving medium has been started in relation with the acoustic streaming phenomena [72–74]. The most popular method for deriving acoustic equations has been the perturbation method that was originated in the pioneering papers by Helmholtz, Blokhintsev [75] and Eckart [72]. It is based on perturbation evaluation of every basic fields, like, for instance, density:

$$\rho = \rho_0 + \rho_1 \in + \rho_2 \in^2 + \dots \quad (118)$$

In 1963, a new, more correct, approach to formulation of the basic acoustic equations was formulated simultaneously by Biot [76] and Eckart [55]. In this new approach the acoustic motion is viewed as a small superimposed motion on a given fluid flow under the initial stress and initial temperature. Let us consider two moving fluids or two modes (excited and mean) of the motion of the same fluid. Let both of them differ slightly only by a small displacement $\vec{\xi}$. Such a situation can be called ‘small superimposed motion’. Various concepts of superpositioning of two motions are shown in Fig. 3. The comparison of these two motions may be made in several ways—two of the simplest ways are discussed below.

If both motions ϕ and ϕ' are specified in the Lagrangian material coordinates (Fig. 3b) then a one-to-one correspondence can be set up between their particles by associating particles with the same values of \vec{X} . In this way the two flows may be compared using the \vec{x} configuration and the description of superimposed motion by the following displacement vector $\vec{\xi} = \vec{\xi}(\vec{x}, t)$. However, if ϕ and ϕ' are specified in the Eulerian spatial coordinates then a fluid particle that actually flows through position \vec{x} (Fig. 3a) can run from two, slightly different positions \vec{X} and \vec{X}^ξ , which differ by a displacement vector $\vec{\xi} = \vec{\xi}(\vec{X}, \tau)$.

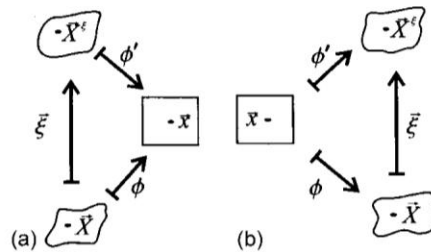


Fig. 3. Two possible approaches for comparison of the excited motion superimposed on a normal motion of fluid element.

Note that the first picture (Fig. 3a) is consistent with the Eulerian description and the second one (Fig. 3b) is consistent with the Lagrangian description that, as a result, leads to the Lagrangian mean approach. This last treatment has also been used in the description of water waves in oceans [45,47,56], turbulence [77] and electrodynamics [78,79]. If fruitfully happens that both cited papers of Biot [76] and Eckart [55] are complementary. Biot's paper concentrates on the question of what happens with internal energy ε in superimposed motion. However, Eckart concentrates on what happens with the kinetic energy in superimposed motion.

The complete reconstruction of the Biot–Eckart approach needs more attention and will be presented in a separate paper [80].

10. Generalized Lagrangian Mean approach

There are three general approaches to the formulation of turbulent motion equation that would include a mode of turbulent momentum flux. The older approach, which has its roots in the kinetic theory of gases, was formulated and developed by Reynolds [31]. Reynolds' approach is also known as the 'splitting' treatment that postulates a splitting of an arbitrary state of fluid into 'mean' and 'excited' ones. Then, any particle of fluid is composed of two subfluids that flow with different velocities, temperatures, pressures, entropies and so on. Reynolds postulated an internal transport of mass, momentum and energy between both subfluids via additional turbulent fluxes of mass, momentum and energy. These additional turbulent fluxes must be postulated as independent equations. In the original Reynolds approach there is no averaging in space or time—there is only the splitting. For instance, the turbulent flux of momentum, proposed and developed by Reynolds ([31], Eq. (222)) has, in the contemporary notation, a form [81]:

$$R_{ij} = \left(1 + \frac{\sigma}{\lambda}\right) \left\{ \frac{\rho}{2} k \delta_{ij} + \frac{1}{2} \frac{\lambda k}{\sqrt{\pi}} [(\rho J_i^t - \rho v_i)_{,j} + (\rho J_j^t - \rho v_j)_{,i}] \right\} \\ + \frac{\rho}{2} [(J_i^t + v_i)k_{,j} + (J_j^t + v_j)k_{,i}], \quad (119)$$

where λ is the mixing length, σ —the area of turbulent contact, \vec{J}^t —turbulent flux of mass. This flux depends only on one additional parameter k that has a meaning of kinetic turbulence energy. The model can be called the 'gradient model of turbulence'.

The second approach is based on the procedure of the Eulerian averaging in time. It is traditionally connected with the name of W.C. Reynolds. In practice, this approach is known as 'RANS' Reynolds-Averaged Navier–Stokes' [82,83]. The basic mathematical procedure on which the RANS is based, is an averaging of the momentum equation.

The third approach, introduced by Andrews and McIntyre [17] is based on the Lagrangian averaging procedure performed not on the momentum balance but on the pseudomomentum balance. This approach is known as the Generalized Lagrangian Mean (GLM), or, as proposed recently by Holm [19]—'LANS'—Lagrangian Averaged Navier–Stokes.

Atmospheric and ocean motions are strongly affected by waves that generate rectified effects and the 'mean' motion [84]. This is the case with the Langmuire circulations occurring in the oceans. When the wind blows over a water surface and generates waves,

numerous streaks parallel to the wind direction may be observed. Langmuire related these streaks to convergence lines between counter-rotating vortices below the surface. Different mechanisms for generating Langmuire circulations, proposed in the literature, are based on the postulate that these vortices are induced by interactions between the wave field and mean wind-induced shear flow. A motivation for the Andrews and McIntyre model introduction was Leibovich's [85] opinion that interactions between wind-generated waves and currents are essentially better described within the pseudomomentum framework.

It should be noted that the Generalized Lagrangian Mean model describes mean motions and is therefore conceptually equivalent to Euler averaging or W.C. Reynolds averaging. However, practically, GLM describes Lagrangian aspects (pseudomomentum) in the Eulerian setting. The starting point of GLM is Fig. 3b where wave displacement $\vec{\xi} = \vec{\xi}(\vec{x}, t)$ is measured from the observer's current position. The Lagrangian mean averaging in time $[\cdot]''$ invokes any averaging operation that commutes with time or space differentiation and requires:

$$[\vec{\xi}(\vec{x}, t)]'' = 0. \quad (120)$$

This defines a particular reference trajectory, one for which the average wave displacement from the reference path of any quantity evaluated at the true Lagrangian position of a fluid particle is zero. This would seem to be a natural definition of 'mean Lagrangian' motion and the mean Lagrangian velocity \vec{v}^L . This leads to the pseudomaterial derivative [17]

$$\frac{D^L}{Dt}(\bullet) \equiv \frac{\partial}{\partial t}(\bullet) + [(\bullet) \otimes \nabla] \vec{v}^L. \quad (121)$$

The resulting equation of pseudomomentum balance is as follows [17,86]:

$$\frac{D^L}{Dt}(\vec{v}^L - \vec{u}^L) + (\vec{v}^L - \vec{u}^L)(\vec{v}^L \otimes \nabla) = -\nabla \pi + \vec{F}^L, \quad (122)$$

where the pseudomomentum (per unit mass), similar to Eckart's definition (64) is [17]:

$$\vec{u}^L = -[\xi_{j,i} \xi_{j,t}]'' \vec{e}_j \quad (123)$$

and

$$\pi = p^L / \rho^\xi + 1/2 [\xi_{i,t} \xi_{i,t}]'' \quad (124)$$

is the spherical pseudomomentum flux, whereas \vec{F}^L is the viscous contribution to pseudomomentum.

The Eqs. (122) together with the definitions for \vec{F}^L govern the Lagrangian mean fields. Let us note that the role of the Reynolds turbulence flux now is being played by a single vector of pseudomomentum \vec{u}^L for which an arbitrary closure must be postulated. This suggests that the closure requires only three terms, versus six components of the Reynolds flux of momentum. Thus, there is an analogy with the Broszko model of rational turbulence [81].

11. Conclusions

Most of the potential advantages of the discussed pseudomomentum balance are yet to be explored, so we confine ourselves to just a few remarks. To illustrate the potentials of the pseudomomentum notion we have re-evaluated the original approaches proposed by Thomson and Tait, Gibbs, Natanson and Eckart. Fortunately, it follows from our investigations that the variational approach is more useful in the cases of new, unrecognized physical phenomena. Let us note that the role of variational approaches in the proper examination of pseudomomentum flux is quite the same as in the case of the variational approach to the proper definition of the momentum flux in so-called nonsimple continua [4,34,54,87,88]. This remark is also in agreement with the case of a proper definitions of momentum and pseudomomentum within the framework of the extended irreversible thermodynamics. In that case, a line of reasoning, proposed by Sieniutycz [89], could be adopted and developed.

In our opinion, the set of the pseudomomentum balances in fluids, studied in this chapter does not exhaust the potential of the configurational force approach. Along with new applications in acoustics, turbulence and wave thermomechanics, the extension of the discussed notions with viscous and conductivity effects, is an interesting and, we believe, realistic prospect.

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